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Investigations on Proof-Theoretic Semantics

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Dissertação apresentada ao Programa de Pós-Graduação em Filosofia da Universidade Federal de Goiás, como requisito parcial para obtenção do título de Mestre em Filosofia.

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Resumo

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As semânticas construtivas oferecem uma nova abordagem semântica para as constantes lógicas. Essas semânticas gozam de fortes motivações filosóficas advindas da filosofia da linguagem e da filosofia da matemática. Nós investigamos essa nova abordagem semântica da lógica e sua concepção de validade lógica sob a luz de suas próprias aspirações filosóficas, em especial aquelas representadas pelo trabalho de Dummett (1991). Dentre nossos resultados, destacamos a validade da Regra de Peirce em relação ao procedimento justificatório baseado nas regras de introdução para as constantes lógicas proposicionais. Essa é uma situação indesejável, pois a Regra de Peirce não é considerada aceitável de um ponto de vista construtivo. Por outro lado, verificamos que o procedimento justificatório baseado nas regras de eliminação atesta a invalidade dessa mesma regra. Tecemos alguns comentários a respeito das consequências desse cenário para o projeto filosófico de Dummett e para as semânticas construtivas em geral.

Palavras-chave intuicionismo lógico, teoria das demonstrações, teoria do significado, validade lógica

Abstract

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Proof-theoretic Semantics provides a new approach to the semantics of logical constants. It has compelling philosophical motivations which are rooted deeply in the philosophy of language and the philosophy of mathematics. We investigate this new approach of logical semantics and its perspective on logical validity in the light of its own philosophical aspirations, especially as represented by the work of Dummett (1991). Among our findings, we single out the validity of Peirce's rule with respect to a justification procedure based on the introduction rules for the propositional logical constants. This is an undesirable outcome since Peirce's rule is not considered to be constructively acceptable. On the other hand, we also establish the invalidity of the same inference rule with respect to a justification procedure based on the elimination rules for the propositional logical constants. We comment on the implications of this scenario to Dummett's philosophical programme and to proof-theoretic semantics in general.

Keywords logical intuitionism, proof theory, meaning theory, logical validity

Intuitionism is a scandal to those who think that philosophy is of no importance, or that it cannot affect anything outside itself, or at least that there are some things which are sacrosanct and beyond the reach of philosophy to meddle with, and that among them are the accepted practices of mathematicians.

Elements of Intuitionism
Michael Dummett

Contents

Introduction

Validity is an important concept in logical investigations. If we understand logic as the study of reasoning and argumentation, for instance, then there is no doubt that validity is a *necessary* condition of correct reasoning and cogent argumentation. As a result, throughout the history of logic, the notion of validity has always played a central role.

Validity is a semantic concept. This means that, to show validity of an argument or class of arguments, we must take into account the *meanings* of the logical expressions involved. It is by appeal to the meanings of the logical expressions that we can argue for the correctness of an inference from premisses to conclusion in a valid argument. Therefore, the general model used to explain meaning, i. e., a *meaning theory*, has a fundamental impact on the corresponding concept of validity.

Currently, the prevalent meaning theory is *denotational* meaning theory: a theory which explains meaning on the basis of *reference* or *denotation*. As a result, most students of logic are acquainted with the concepts of *interpretation*, *valuation*, *satisfaction*, *truth value* and others notions associated with model-theoretic semantics, i. e., a particular kind of denotational semantics that borrows much of its technical notions from model theory, a branch of mathematics.

The development of model-theoretic semantics represented a notable change of attitude in logical investigations. This may be hard to see because many logic textbooks, following a common trend in the field, presents model-theoretic semantics as a natural extension of the early developments of modern symbolic logic. However, there was indeed a drastic change of approach to the explanation of logical validity. The distinction between syntax and semantics, for instance, was absent in the early days of modern symbolic logic. Thus, although Frege and Russell made heavy use of symbolic notation, the symbols on their symbolic languages were never intended to be dissociated from their meanings: their formal languages were presented as a *notation* for expressing logical notions and relations. Looking at their work, we can see how they took great care, when introducing logical notions and relations, to explain their meaning by means of examples and by describing their general behavior.

It was David Hilbert who took the first important steps towards the separation of syntax from semantics when he proposed, for the sake of pursuing his consistency proof, that we view the symbolic systems of Russell and Frege as syntactical systems. In this way, symbolic logic started to change from logical investigations made more precise with the *use* of symbols to investigations *about* the symbolic systems themselves. A radical and interesting example of this kind of formalism can be found on Carnap (1964, p. 1):

> The prevalent opinion is that syntax and logic, in spite of some points of contact between them, are fundamentally theories of a very different type. [...] But the development of logic during the past ten years has shown clearly that it can only be studied with any degree of accuracy when it is based, not on judgments (thoughts, or the content of thoughts) but rather on linguistic expressions, of which sentences are the most important, because only for them is it possible to lay down sharply defined rules.

In *The Logical Syntax of Language*, Carnap attempted to answer traditional logical problems by developing his theory of pure logical syntax. Later, Tarski (1956) introduced the notion of *model*[1] in order to overcome what he saw as shortcomings of the "syntactic approach to logical consequence". After Tarski's work, model-theoretic semantics sided with the (syntactic) proof theory of the Hilbert school and became an indispensable part in modern logical theories. Logic became twofold: syntax and semantics.

With model theory taking care of the semantics, the common conception is that the syntax ought to be understood as pure combination of symbols. The familiar recursive definition of the set of well-formed formulas, for example, is a well-known part of the syntax. But in modern logical theories, the syntax does not concern itself merely with the construction (or specification) of formal languages. The *formal proofs* of a deductive system, especially Hilbert-style deductive systems, are understood as transformations and operations on strings of symbols completely devoid of meaning and thus are also a part of syntax. However, just as the grammar of the formal language is based on the semantic role of its syntactic units[2], the formulation of the rules for carrying out formal proofs always has also the meaning in sight.

[1]It is important to notice that the notion of *model* as originally used by Tarski (1956) differs substantially from the notion of model currently used in model theory. There is no doubt, however, that Tarski's work was the most important inspiration.

[2]For instance, in a formal grammar, the symbols of the formal language are classified into

Notwithstanding, model-theoretic logical theories generally assume a formalist stance towards syntax and proceed as if it was independent from the semantics, i. e., from meaning. As a result, one can feel an implicit dichotomy between syntax and semantics which gives rise to the belief that syntax is somehow opposed to meaning. But is formalism the only approach to syntax? Is it correct that rules and rule-following are essentially opposed to meaning?

As remarked above, some important pioneers of modern symbolic logic did not take a formalist stance towards their symbolic systems. Moreover, judging from some pieces of his work, even Hilbert (1928, p. 79), by some considered to be the father of formalism, considered formal proofs as an expression, or representation, of meaningful thought:

> The formula game that Brouwer so deprecates has, besides its mathematical value, an important general philosophical significance. For this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed.

In other words, Hilbert is saying that the syntactic rules used to construct formal proofs are not mere symbolic manipulation but also an expression of deductive relations.

Besides its contribution to the adoption of a formalist approach to syntax, model-theoretic semantics also changed significantly the way we establish validity. In ancient Greece, Aristotle (Prior Analytics, 29b) argued intuitively for the validity of some deductive relations and later established the validity of a group of other deductive forms by reducing them to the first ones. In contrast, model-theoretic semantics explains validity in terms of quantification over models and, as a result, all valid forms are, conceptually, on the same level[3].

In other words, the pervasiveness of model-theoretic semantics and its syntax/semantics dichotomy has shifted the attention of the logician away from inferences and deductions and has placed it instead into valuations and models. This has gone to such an extent that one needs to write a handful of paragraphs just to explain how inferences, as steps in a proof or argument, can ascribe meaning

classes that indicate their general semantic role — individual constants are used to denote objects of the domain, monadic predicate letters denote subsets of the domain whose elements are objects that have a certain property and so on.

[3]For example, from the viewpoint of the tarskian notion of logical consequence, there is no conceptual difference between the validity of *modus ponens* and that of an argument form with an infinite class K of premisses.

to logical expressions and how proof theory, as the study of such deductive practice, can be a base for semantics. As Schroeder-Heister (2006, p. 526) adequately observes:

> Proof-theoretic semantics [...] uses ideas from proof theory as a mathematical discipline, similar to the way truth-condition semantics relies on model theory. However, just this is the basis of a fundamental misunderstanding of proof-theoretic semantics. To a great extent, the development of mathematical proof theory has been dominated by the formalist reading of Hilbert's program as dealing with formal proofs exclusively, in contradistinction to model theory as concerned with the (denotational) meaning of expressions. This dichotomy has entered many textbooks of logic in which "semantics" means model-theoretic semantics and "proof theory" denotes the proof theory of formal systems. The result is that "proof-theoretic semantics" sounds like a contradiction in terms even today.

The main theme of this dissertation is proof-theoretic semantics. It is based on the idea that meaning should be explained not in terms of *denotation* but in terms of *use*. To explain meaning in terms of use in the context of a semantics for logic is to adopt the view that certain deductive rules implicit in our linguistic practice determine the meaning of the logical constants. Thus, proof-theoretic semantics is to a meaning theory based on use what model-theoretic semantics is to a meaning theory based on denotation.

We can better illustrate the proof-theoretic approach to meaning by means of an example with a single logical constant: implication. Since we are concerned with a semantics for logic, the relevant practice is *deductive* practice; the relevant use is *deductive* use. There are two aspects to the use of implications in deductions: they can appear as conclusion or as premisses of an inference step.

However, there are many ways in which implications can appear as either premiss or conclusion of inferences, but not all of them are essential to the meaning of implication. We observe that there are essential, canonical uses of implication either as premiss or as conclusion of an inference. When used as the conclusion, we can express the canonical uses by especifying the necessary and sufficient conditions for concluding a sentence with implication as the main logical connective.

We can read the introduction rule of natural deduction below as expressing exactly these necessary and sufficient conditions. Thus, a necessary and sufficient condition for making an inference whose conclusion is $A \supset B$ is that we have a derivation of B from the hypothesis A. Of course, there are other situations in which $A \supset B$ may appear as conclusion of an inference. But, as we shall be convinced later (Section 1.6), these other uses are inessential and can be explained by reference to the canonical use.

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B} \tag{1}$$

Likewise, the elimination rule for implication below can be seem as the canonical way to infer consequences from sentences with implication as their main logical connective. Here, the elimination rule expresses what consequences must be accepted on the strength of $A \supset B$ and the auxilliary premiss A. Again, there are other consequences that can be extracted from $A \supset B$ besides those of the corresponding elimination rule. And, again, they are inessential.

$$\frac{A \quad A \supset B}{B} \tag{2}$$

By carrying the considerations sketched above to the other logical constants, we can show that any valid deductions can be accounted for by reference only to canonical inferences. In other words, any deductive relatioship between sentences can be established using the introduction and eliminations rules of the logical constants involved.

Furthermore, we can notice an important relationship between both aspects of the use of implication: what was required for the introduction of $A \supset B$, namely, a derivarion of B (based on hypothesis A) can be restored by applying the elimination rule. To put it in another way, what is obtained by elimination of $A \supset B$ was already at hand if we assume $A \supset B$ to have been derived by the introduction rule.

The relationship between the deductive behavior of introduction and elimination rules for a logical constant can be studied in order to extract important semantic properties. A general study of this kind constitutes the core of the proof-theoretic approach to semantics[4]. Our little example using implication was designed only to illustrate, somewhat roughly, the proof-theoretic approach to the semantics of logic. A more substantial discussion requires a more detailed treatment of natural deduction systems and some of their properties (see Chapter 1).

[4]Some decades ago, Prawitz (1973) outlined a research program along the same lines which he called "general proof theory" in constrast with the "reductive proof theory" of the Hilbert school. Later, Schroeder-Heister (see 2006) proposed the term "proof-theoretic semantics" which is widely adopted today.

One of the challenges facing proof-theoretic semantics is to determine to what extent we can achieve a satisfactory logical theory, including a well-defined conception of validity, by reflection on the deductive use of the logical constants. However, the interest placed on proof-theoretic semantics does not come from logic alone, but also from mathematics. On the literature, proof-theoretic semantics has been associated with intuitionism and mathematical constructivism generally. In particular, Dummett (1975b) believes that we can settle the metaphysical dispute in the philosophy of mathematics by advancing arguments in favour of a meaning theory based on use, which in turn will lead solely to a justification of intuitionistic reasonings and, thereby, a vindication of the intuitionistic philosophy of mathematics (see Chapter 2). Thus, he maintains there is a path from logic to metaphysics such that, by resolving the dispute between classic and intuitinist in the philosophy of logic, we also resolve the dispute in the philosophy of mathematics.

With respect to the resolution strategy proposed by Dummett, we can doubt whether the conflict in the philosophy of mathematics does not involve strictly *mathematical* (non-logical) considerations but merely revolves around what is the correct underlying logic[5]. However, as we shall see (Section 2.2.2), from a philosophical point of view, a more serious problem afflicts some current proof-theoretic proposals: the restriction of semantic analysis to the concept of *proof* which, altough an important part, is not sufficient to account for deductive practice (in mathematics and other areas of discourse).

It is quite evident from our practice that we can make deductions from explicit (open) assumptions. Yet, many constructive accounts of logic adopt what Schroeder-Heister (2013, Section 2.2.2) calls "the substitutional view of open proofs". According to this view, proofs (deductions) from opens assumptions must be explained away in terms of proofs *tout court*.

As we shall discuss in Chapter 2, the substitutional view of open proofs is based on some general philosophical tenets which, altough persistent in much of the discussion around proof-theoretic semantics, can be challenged from an autentic proof-theoretic perspective. In particular, we single out the following two ideas as important obstacles to the acceptance of deduction from assumptions as a primitive concept:

assertability The idea that assertions are the central linguistic concept of a mean-

[5]Philosophical views associated with mathematical constructivism can be very diverse and rich. They sometimes involve positions with regard to some strictly mathematical concepts. Troelstra and van Dalen (Section 1.4, 1988) offers a concise survey of the most important philosophical positions associated with constructivism in 20th century mathematics.

ing theory based on use and, consequently, the view that an inference is a passage from assertions to assertions. This is often expressed in the slogan that a meaning theory based on use substitutes, in the general framework of a denotational meaning theory, the concept of *truth conditions* for *assertability conditions*.

BHK semantics The idea that the BHK interpretation of the logical constants should be taken as a starting point for a complete and coherent semantic explanation of the meaning of the logical constants and, in particular, the view that the BHK interpretation corresponds somehow to the introduction rules of natural deduction.

Once these ideas are abandoned, there is no reason to cling onto a substitutional view of open proofs. Then, much as we have been doing, we can, consistently with our proof-theoretic semantic principles, understand the meaning of the logical constants as they are expressed in our practice of deductive argumentation. In particular, if one of the aspects of the meaning of a logical constant is indeed expressed by its use in open assumptions so that we can extract consequences from them, then there seems to be no overwhelming reason why we should reduce this aspect to a notion of categorical proof (proof from no assumptions).

We believe that our diffidence towards the substitutional view of open proofs would be justified in Chapter 3, when we examine the proof-theoretic semantics proposed by Dummett (1991, Chapter 11–13). There we provide Theorem 1 which states the validity of Peirce's rule with respect to Dummett's justification procedure based on introduction rules.

Chapter 1

Natural Deduction

In what follows, we shall strive to bring out semantic content from natural deduction[1]. In line with this motivation, the following discussion does not contain an exaustive or self-contained treatment. Complete, excellent and rigourous technical monographs on natural deduction are available elsewhere and very little could be achievied by reviewing them here. Altough we will, indeed, review the introduction and elimination rules for the usual logical constants, our focus will be on *explanation* rather than specification and presentation. However, before we proceed, it is necessary to quickly fix some notation to avoid confusion.

We use the symbols "$\supset$", "$\wedge$", "$\vee$", "$\neg$", "$\forall$" and "$\exists$" to mean *implication*, *conjunction*, *disjunction* and *negation*, *universal quantification* and *existential quantification* respectively. Uppercase latin letters "A","B", "C" and so on stand for arbitrary sentences (open or closed). We use lowercase greek letters, like "φ" and "ψ", specially to denote atomic or prime sentences. Uppercase greek letters "Γ" and "Δ" stand for sets of sentences which are normaly used to represent the premisses on which depends some occurence of a sentence in a derivation. The derivations themselves are depicted as trees of sentences where we indicate discharche of hypotheses by enclosing them in square brackets as in "[A]". The uppercase greek letter "Π" is reserved for trees or subtrees and the uppercase greek letter "Σ" is used to denote sequences of trees (including an empty one) which can be part of a complete tree. Moreover, some uppercase latin letters (in sans serif typeface) are reserved to denote sets of rules in natural deduction style as in "I" for introduction rules and "E" for elimination rules. Among the lowercase latin letters, we use "a", "b" and "c" as individual parameters; individual variables are denoted by "x" and "y"; and, finally, we reserve "t" for terms. When necessary,

[1]The idea is not new and can be traced back to the "gentzensemantik" of Kutschera (1968).

we employ natural number subscripts.

A natural deduction system consists of a set of rules designed to capture the concept of logical deduction. The most interesting feature of natural deduction compared to other deductive systems is the classification of its inference rules between introduction rules and elimination rules (at least one of each kind for each one of the logical constants). Natural deduction rules for a certain logical constant always figure a formula with that constant as the main operator and also its subformulas. Moreover, as a general pattern, the subformulas occur as premisses in the introduction rules whereas with the elimination rules it is usually the other way around[2].

This general pattern naturally gives rise to the interpretation that the introduction rule for a logical constant γ, denoted by "γI", expresses the necessary and sufficient conditions under which we can infer a sentence containing γ as the main logical operator. Analogously, the elimination rule for γ, denoted by "γE", expresses what are the consequences that can be extracted from a sentence containing γ as the main logical operator, together with other, minor, premisses when necessary. Below, we show the introduction and elimination rules for the main logical constants. We also take the opportunity to make some comments about their meaning.

1.1 Implication

Also called "conditional", this connective is perhaps the most complicated and controversial of all the logical constants. Its most common English reading is represented by the "if ... then" idiom. In general, when we use this expression, we claim a certain relation of entailment between the antecedent and the consequent: one follow from the other by causality, deduction or other kind of chain of plausible reasons. However, the meaning attached to implication by the introduction rule is somewhat weaker than the meaning usually associated with the "if ... then" idiom.

[2]In fact, the situation with the elimination rules is a little more complicated. The formula containing the logical constant as main operator occur as a *major premiss* with subformulas sometimes also figuring as *minor premisses*. A subformula may (as in inference rules) or may not (as in deduction rules) occur as conclusion of the rule.

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B}\supset\text{I} \qquad\qquad \dfrac{A \quad A \supset B}{B}\supset\text{E}$$

As a matter of fact, the introduction rule above *does not require* that a premiss of the form A be actually discharged. Thus, the rule permits the inference of $A \supset B$ on conditions under which A is irrelevant. On the other hand, the elimination rule for implication *does require* a premiss of the form A for its application. From the point of view of the introduction rule, the stronger requirement for elimination is natural since, even assuming the major premiss to have been obtained by $\supset$I, we cannot tell in advance if a premiss was discharged by its application. However, from the point of view of the elimination rule, a stronger meaning can be assigned to implication since A will always be available but would go unused in a deduction of B not depending on A.

The unbalance between the introduction and elimination rules for implication and the disagreement between those rules and the expressions said to be its equivalent in natural languages has led some authors, notably Tennant (1987, chapter 17), to favor a relevant reading of implication.

1.2 Conjunction

Especially in contrast with implication, the rules for conjunction are perhaps the most straightforward and uncontroversial. This is no surprise, since the meaning that the introduction and elimination rules give to conjunction is very narrow.

$$\dfrac{A \quad B}{A \wedge B}\wedge\text{I} \qquad\qquad \dfrac{A \wedge B}{A}\wedge\text{E} \qquad\qquad \dfrac{A \wedge B}{B}\wedge\text{E}$$

A simple, although not so exact, intuitive explanation of the rules is to say that a conjunction allows one to convey, in a *single* sentence of the form $A \wedge B$, the exact same information conveyed by both sentences, A and B. This narrow meaning is seldom, if ever, intended in ordinary speech where expressions like “and” and “but” are more often used to convey more information than mere logical conjunction: temporal sequence, surprise, disbelief and so on.

1.3 Disjunction

In English, disjunction is usually associated with the meaning of the expression "or". In ordinary speech, we sometimes use "or" to express an exclusive choice or option as in "you either pay your rent or risk having to sleep under the bridge". Thus, it is common to use "or" in contexts involving agents and actions. Yet, disjunction also appears in more declarative contexts, especially in situations when we do not have enough information to determine which one of the disjuncts holds. Even in such state of information, we can still extract consequences from the disjunction by showing them to be derivable from each one of the disjuncts, as is expressed by the elimination rule.

$$\frac{A}{A \vee B}\ \vee\text{I} \qquad \frac{B}{A \vee B}\ \vee\text{I} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\ \vee\text{E}$$

On the other hand, at least from an epistemological point of view, the introduction rule for disjunction is somewhat pointless. As remarked above, the conclusion $A \vee B$ does not say which one of the disjuncts holds. So, epistemologically, there seems to be more information on the premiss of the rule than on its conclusion. Arguably, this can also be said of $\wedge$E. But, given that $\wedge$E is an elimination rule and its purpose is to extract consequences from its premiss, it is expected that those consequences may have less information.

1.4 Negation

In Prawitz (1965) and in most of the modern texts in natural deduction, negation is a defined symbol. It is defined in terms of a propositional constant $\bot$, called "absurdity" or "falsum", and implication: $\neg A \equiv A \supset \bot$. A noteworthy exception is Gentzen (1935, p. 186), which gives introduction and elimination rules for negation. We follow the current practice and give rules for the $\bot$ constant, letting negation stand defined as above.

$$\frac{\bot}{A}$$

Negation is also a very interesting and controversial logical operator. The classical understanding of negation goes back to Aristotle (On Interpretation, VI) which believed that every sentence had its corresponding contradictory negation which could, in principle, be affirmed. As a matter of fact, from the point of view of a denotational theory of meaning, this Aristotelian conception of negation seems very natural.

However, if we choose to remain silent about Aristotle's claim and try to come up with some general rules about negation, one of the principles that comes to mind is the so called law of non-contradiction. The principle of non-contradiction is often understood as a prohibition (among rational human beings) of mantaining both A and $\neg A$ at the same time. Using the rule for $\perp$ and the definition of negation above, we can get a sense of the consequences of violating the principle of non-contradiction: any sentence can be obtained, and our deductive practice looses all its meaning.

Whether our rule for $\perp$ is the best way to account for our use of negation, even in mathematical discourse, is indeed open to question. Nevertheless, there is a far more troublesome issue with our account of negation: $\perp$ does not follow the pattern of introduction and elimination rules laid down by the other constants. In order to remedy the situation, Dummett (1991, p. 295) has proposed the following introduction rule for $\perp$, where A_i ranges through all the atomic sentences of the language:

$$\frac{A_1 \quad A_2 \quad A_3 \quad \dots}{\perp}\ \perp\mathrm{I}$$

When the natural deduction rule for $\perp$ is considered as an elimination rule, written "$\perp$E", Dummett believes that the rule $\perp$I above is the more adequate harmonious introduction rule. Indeed, we can see that, assuming A to be atomic without loss of generality, an application of $\perp$E would allow us to obtain any atomic sentence A. Therefore, an introduction rule in harmony with $\perp$E should require no less than all atomic sentences A_i as premisses (see Section 1.6).

Now, someone might wonder why the rule $\perp/A$ is sufficient for a complete natural deduction system while the other constants require introduction *and* elimination rules. As Dummett (1991, p. 292) himself observes, on the face of the catastrophic consequences of actually accepting an argument for $\perp$, we can expect to use it only in subordinate arguments. For instance, we can use it to show the unacceptability of some claim A by means of an argument from A to $\perp$ and, subsequently, conclude $A \supset \perp$ and discharge the unacceptable hypotheses A. At any rate, the meaning of negation, as expressed in natural deduction systems with our rule for $\bot$, does not seems to amount to much more than these kinds of subordinate arguments.

1.5 Quantification

Since Aristotle and his syllogistic theory of categorical sentences, quantification is intrinsically connected with predication. It deals with the logical relationship between expression like "some", "every" and "all" when applied to predicates in order to form quantified sentences. Also, it relates quantified sentences with their corresponding singular sentences. In modern logic, the reasonings involving quantified sentences are analysed with the help of two logical constants: the universal and the existential quantifiers. The rules governing the use of these logical constants in a natural deduction system are distinguished by the restrictions on the occurence of individual parameters placed on their applications. In the figures below, we write "$A^{t_1}_{t_2}$" to represent the operation of replacing every occurence of t_1 by t_2 in A (if there are any).

$$\frac{A}{\forall x A^a_x}\ \forall\mathrm{I} \qquad \frac{\forall x A}{A^x_t}\ \forall\mathrm{E} \qquad \frac{A^x_t}{\exists x A}\ \exists\mathrm{I} \qquad \frac{\exists x A \quad \begin{matrix}[A^x_a] \\ \vdots \\ B\end{matrix}}{B}\ \exists\mathrm{E}$$

The meanings of $\exists$I and $\forall$E are very straightforward. The first one licenses the inference of the existentially quantified sentence on the ground that the predicate (or open sentence) A applies to some specific term t. The second one, licenses the inference of a particular instance from a universally quantified sentence. On the other hand, $\forall$I and $\exists$E are more difficult to understand because of the aforementioned restrictions on individual parameters.

In fact, the restrictions are meant to guarantee that we have a general argument concerning an unspecified individual a such that any substitution of a term or individual constant for a, thus obtaining an instance of the general argument, will not tamper with its correctness. Otherwise, if we assume a to occur in one of the premisses on which A depends, the application of $\forall$I does not produce a general argument since the process of instantiation just described will inevitably change some of the premisses. Therefore, in applications of $\forall$I, a must not occur in any of the premisses on which A depends.

Futhermore, a similar general argument is required for $\exists$E because, although we know from $\exists x A$ that we can correctly claim A of some individuals (assuming x to occur in A), we do not know anything specific about them. Therefore, in applications of $\exists$E, a must not occur in any premisses other than the hypotheses discharged, so that the inference to conclusion B does not assume any specific feature of the individual that $\exists xA$ claims to exist.

1.6 Harmony

In the previous section, we saw that we can read the introduction rules for a constant γ as an expression of the necessary and sufficient conditions for deducing its conclusion $A\gamma B$ (assuming the paradigmatic case where γ is a binary connective). Since we are dealing with *logical* expressions, there is a very plausible requirement that we can place on the corresponding elimination rules: the consequences extracted from its major premiss $A\gamma B$ can never extrapolate what was necessary for the conclusion of $A\gamma B$ by means of the introduction rules. A similar requirement can be placed on the introduction rules from the point of view of the elimination rules: given the context, whatever can be deduced from the conclusion by means of the elimination rule could already be deduced from the premisses. When both these requirements are fullfiled, we say that the introduction and elimination rules for a logical constant are in *harmony* with each other.

The concept of harmony between logical rules goes back to a much quoted passage from Gentzen (1935, p. 189) to the effect that "the introduction rules are definitions and the eliminations are only their consequences thereof". Adopting terminology from Lorenzen (1969, p. 30), Prawitz (1965) attempted to make Gentzen's remarks more precise by formulating an inversion principle. Also, harmony is a fundamental part of normalization procedures for natural deduction systems. These procedures show the existence of derivations, called *normal derivations*, with a special structure and important properties. In addition, the normalization theorem establishes that if a sentence A is at all derivable from premisses Γ, then there is a normal derivation of A from Γ.

Normalization procedures are based on *reductions* which allow for the elimination of roundabouts in a natural deduction derivation. In other words, when we have an introduction rule whose conclusion is the major premiss of an elimination rule, there is a reduction which gives us a derivation of the same conclusion from the same premisses without going through those steps. An example:

$$\dfrac{\begin{matrix}\Sigma_1\\A\end{matrix}\qquad\dfrac{\begin{matrix}[A]\\\Sigma_2\\B\end{matrix}}{A\supset B}}{\begin{matrix}B\\\Sigma_3\end{matrix}}\qquad\qquad\qquad\begin{matrix}\Sigma_1\\A\\\Sigma_2\\B\\\Sigma_3\end{matrix}$$

On the left, there is a derivation containing a roundabout: an implication is introduced just to be, immediately after, eliminated. Since $\supset$E is in harmony with $\supset$I, its application just restored what was already required as premiss for the corresponding introduction rule. As a consequence, both steps in the derivation can be avoided by rearranging the derivation as shown on the right.

Besides the possibility of being reduced in the manner described above, there is another interesting property we can, in general, expect from harmonious rules: under certain conditions, their addition to a deductive system yields a conservative extension. For, suppose the conditions for the application of the introduction rule for the newly added connective were fullfiled. Then, if harmony obtains, the elimination rule would not allow the derivation of new consequences besides those that were already derivable in the original system.

The fact that the addition of some rules to a deductive system, or, for that matter, to any comprehensive and coherent linguistic practice, yields a conservative extension makes a strong case for the *logicality* of those rules. Otherwise, if the addition of rules for the use of a logical constant γ change the original system in such a way that a sentence A, not containing γ, now becomes derivable, then we have strong evidence that γ incorporates some extralogical content. This point can be more easily seen when we consider an original system composed solely of descriptive expressions. In such a case, the rules for γ would license the derivation of a descriptive (since it does not contain γ) sentence which was not previously derivable.

Based on our discussion so far, we can gather that, from the point of view of a theory of meaning based on use, harmony is, all things considered, an altogether desirable property for logical constants. Futhermore, we also note that, all of the consequences we have been extracting from the harmonious rules of natural deduction were revealed *by reflection on the meaning of the logical constants* as determined by their deductive use: they are, thus, semantical consequences.

Nevertheless, against the whole idea of a semantics based on proof theory, someone might argue that the purpose of a logical semantics is not only to ascertain validity but also to provide a criterion for invalidity, so that the concept of logical consequence is completely determined. But, if we restrict ourselves to deductions, he continues, and cannot appeal to models and interpretations with which to provide counterexamples, how can we show arguments to be invalid?

Surely, pointing out counterexamples is an important method of showing invalidity. Yet, we have to keep in mind that the method afforded by counterexamples is often used to call atention to *argument form*, as in Bolzano's theory of logical consequence. However, in this case, an independent criterion for invalidity is still needed. In other words, by a conterexample to some argument, we mean another, acceptably invalid, argument of the same form (obtained, in Bolzano's theory, by substitution of non-logical expressions). Admittedly, from a denotational point of view, counterexamples are not arguments but are actually models. Then, these models —which provide for the notion of truth used to explain the classical meaning of logical constants— have to be taken as giving an independent criterion. Whether models of reality can really be accepted as given independently of our deductive practice is, however, a matter of much debate. Prawitz (1974, p. 67) and Dummett (1975a), for instance, discuss this problem and how it affects the capacity of model-theretic semantics to adequately explain and elucidate logical validity. Yet, the question of how should proof-theoretic semantics handle invalidity remains unanswered.

From a classical perspective, with its formalist view of deduction, the prospects with regard to this question does not seem very promissing. Nonetheless, the methods employed in proof-theoretic semantics are not limited to a formalist study of proofs. Prawitz (1973, p. 225, emphasis on the original) understood very well the wide range theoretical implications implicit in Gentzen's work when he proposed his general study of proofs:

> In general proof theory, we are — in contrast — interested in understanding the very proofs themselves, i.e., in understanding not only *what* deductive connections hold but also *how* they are established, and we do not impose any special restrictions on the means that may be used in the study of these phenomena.

Manifestly, if we continue our task of extracting semantic content from an analysis of the deductive behaviour of the logical constants, we shall be led to a very plausible approach to invalidity.

We recall that we can specify all legitimate ways to deduce a conclusion $A \gamma B$ with respect to γI by appeal to normalization. Moreover, by appeal to the notion of harmony, we can also determine the correct use of $A \gamma B$ as major premiss of an elimination rule and consequently fix completely the meaning of γ as a logical constant. Similar remarks can be stated with respect to γE. Obviously, we can extrapolate these methods from a single logical constant γ to a set $\mathsf{I}(\mathsf{E})$ of introduction (elimination) rules. As a result, it is possible to display exactly what are the legitimate ways to obtain a given conclusion with respect to a set $\mathsf{I}(\mathsf{E})$ for the logical constants. Finally, invalidity of an argument can be established by showing that the criteria for legitimately infering the conclusion were not met.

We shall reach a better understanding of the process of validation (or invalidation) in Section 3.2 and Section 3.3 when we investigate the verificationist and pragmatist justification procedures, respectively. As we shall see in those sections, for an invalid rule, the justification procedure produces an additional premiss which does not figure in the rule but is, nevertheless, necessary to correctly infer the conclusion.

In the following chapter, we remark upon the motivation behind proof-theoretic semantics to provide an adequate semantics for intuitionistic logic. On the other hand, besides proof-theoretic semantics, there are other semantics for intuitionistic logic. Kripke semantics is, perhaps, the most well-known. However, there are some objections that can be raised against Kripke semantics from a constructive point of view. For instance, it is possible to show validity in Kripke semantics without actually producing a proof (or "witness" as some authors use in this context). With respect to invalidity, Kripke semantics can be motivated via Brouwer's method of weak counterexamples. As Kripke (1965, p. 104) himself remarks:

> A careful reader of the present section on the interpretation of our models will find it plausible that, conversely, a good deal of the interpretation, at least for propositional calculus, that has just been carried out in FC, could be carried out using Brouwer's method of ips depending on the solving of problems.[3]

In a Kripke model[4] a counterexample is represented by a node (state of information) in a Kripke tree such that we accepted the premisses but are, nevertheless, not forced to accept the conclusion. If the conclusion is not refutable, supposedly there are other nodes farther up in the tree (in which we accepted other premisses) such that we are forced to accept the conclusion. Maybe, the additional premisses produced by the proof-theoretic justification procedures can be shown, in a certain sense, to correspond to weak counterexamples and Kripke countermodels.

[3]In "Brouwer's method of ips", Kripke is refering to Brouwer's infinitely proceeding sequences. For a detailed discussion, see Heyting (1971, Section 8.1).

[4]For a complete and clear exposition of Kripke models for intuitionistic logic, see Troelstra and van Dalen (Chapter 2.5, 1988).

Chapter 2

Philosophical Motivations

There are at least two philosophical motivations connected with the development of proof-theoretic semantics:

- To advance an argument for the adoption of intuitionistic over classical logic
- To develop a semantics for logic based on the "meaning as use" approach to the theory of meaning

The first motivation stems from constructivist views on the philosophy of mathematics and seeks to gain support for constructive mathematics by means of replacing classical logic with constructive logic. A very compelling exposition of such motivation was made by Dummett (1975b, p. 5).

The second motivation stems from an anti-realist position in the philosophy of language and seeks to develop an autentic semantics, for logic and other areas of discourse, which is faithful to the view that meaning must be based on use.

Dummett (1991) has also expressed the second motivation. In fact, a noteworthy characteristic of his work is the amalgamation of both motivations into a coherent philosophical programme. According to this programme, considerations from the philosophy of language can be used to settle the metaphysical dispute between classical and constructive philosophies of mathematics.

In outline, Dummett's programme begins at the level of the philosophy of language by advancing arguments against denotational theories of meaning and truth-conditional semantics. Then, an alternative, more adequate, conception of meaning based on use is proposed. The programme culminates in the development of a complete semantics for logic. In addition, there is an expectation that the semantics will avail intuitionistic logic over classical logic and, finally, *through logic*, will settle the controversy in the philosophy of mathematics.

However, as we can see, there is nothing in the formulation of the programme itself that rules out the possibility that a theory of meaning based on use *will not*, after all, favor intuitionistic logic over classical logic. The reasons to expect this outcome rests mainly on Dummett's meaning-theoretical interpretation of the metaphysical dispute between classical and intuitionist mathematics.

Dummett maintains that the metaphysical dispute between classical and intuitionist mathematics is part of a wide range of metaphysical disputes between two general opposing camps: realism and anti-realism. In his interpretation of this class of metaphysical disputes, the difference between the camps boils down to the question of what is the correct theory of meaning for the relevant class of sentences and, in particular, whether the principle of bivalence applies.

The connection between the theory of meaning and metaphysics has often been criticized on general grounds, for instance, by Pagin (1998) and Devitt (1983). In contrast, our concern shall be to investigate the programme on its own terms and see whether the development of proof-theoretic semantics is indeed capable of settling the metaphysical dispute by vindicating only intuitionistic logic.

There is another important issue concerning Dummett's programme that can sensibly be addressed by actually carrying it out: to show that a satisfactory and coherent meaning theory based on use is possible at all. As Dummett (1991, chapter 10) himself observes, not only the mere possibility of such a theory of meaning, but also its capacity to effectively criticize and maybe reform accepted linguistic usage faces important threats from semantic holism.

Given the broad philosophical connections of Dummett's programme, we shall examine some relevant conceptual issues in the next sections before we address the details of a functional proof-theoretic semantics for logic in Chapter 3.

With the objective of getting a better understanding of its constructive heritage, we need to examine some historical roots since much of the conceptual framework of proof-theoretic semantics was a product of the late 19th century (and early 20th century) debate on the foundations of mathematics. Among the many issues debated, we single out two important historical roots. First, there is Brouwer's intuitionistic philosophy of mathematics. And second, there is Gentzen's formalisms and his conceptual analysis of deductive reasoning.

Gentzen's natural deduction was already treated to some extent on Chapter 1. There we saw that Prawitz (1965) expanded on Gentzen's work on natural deduction and formulated what he called an *inversion principle*. This principle expressed an *harmony* between the inferential behaviour of the introduction and elimination rules for a given logical constant. We also saw the importance of *harmony* and how it became a cornerstone for proof-theoretic semantics.

In the next section, we expand on intuitionism. Its main contribution was the critique of classical mathematical reasoning, especially of what is known as the *principle of excluded middle*. This led to the development of the canons of reasoning expressed in intuitionistic logic.

2.1 Logic and mathematics

The late 19th century witnessed a vigorous debate around the foundations of mathematics. As the logicism of Frege, Russell and Whitehead fell prey to paradoxes, two distinct philosophies came out as alternatives to logicism: Hilbert's formalism and Brouwer's intuitionism.

On the one hand, Hilbert's foundational program aimed to show the consistency of mathematics by means of "finitistic" methods. If carried out, Hilbert's consistency proof was believed to provide an indirect foundation for classical mathematics when the more direct approach of the logicists have failed. On the other hand, Brouwer's philosophy rejected any need for foundations: he characterized mathematics as a free product of the mathematician's mental constructions. At that time, the intuitionist's critique of classical mathematics resonated well in the uneasy context of the paradoxes. Despite the fact that, on a practical level, the formalist (in his metamathematics) as well as the intuitionist tried to restrict the principles of reasoning used (if compared to the classical logicist), their philosophies remained quite distinct. As Dummett (2000, p. 2) observes:

> Intuitionism took the fact that classical mathematics appeared to stand in need of justification, not as a challenge to construct such a justification, direct or indirect, but as a sign that something was amiss with classical mathematics. From an intuitionistic standpoint, mathematics, when correctly carried on, would not need any justification from without, a buttress from the side or a foundation from below: it would wear its own justification on its face.

Brouwer's view of mathematics as mentally constructed had drastic consequences to the mathematical practice of his day. An obvious consequence was a rejection of *actual infinity*, a concept that has become widely accepted, especially after Cantor's work on set theory. In this respect, Brouwer was expressing mathematical views which go back, at least, to Gauss (1977–1985) as quoted by Kleene (1952, p. 48).

Apart from the foundational perspective towards mathematics, Brouwer (1908) was also suspicious of logic. He understood clearly that, if the accepeted principles of logic were indeed universally applicable, his views on how mathematics ought to be carried out could not stand unharmed. Thus, he believed that the most elementary constructions of mathematics are not in need of any foundation, logical or otherwise. Rather, he maintained that the logical theories advanced by the logiscists as a foundation for mathematics in fact pressuposed elementary mathematical techniques. Therefore, when Brouwer talked about the principles of logic as being "unreliable", he was attacking the canons of reasoning associated with the predominant classical logic.

2.1.1 The principle of excluded middle

Granted that our main concern is *logic* rather than mathematics, the relevance of Brouwer's philosophy of mathematics to our discussion is that his reflections on the nature of mathematics led to the rejection of what appears to be a purely logical principle. Indeed, he blamed the paradoxes on the careless use of this logical principle, the principle of excluded middle, by mathematicians, especially when reasoning about potentially infinite mathematical series. As Heyting (1971, p. 1) put it:

> It was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction (BROUWER, 1908). The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections.

So, according to Brouwer, one of the main problems with classical mathematics is its uncritical acceptance of the principle of excluded middle when reasoning about potentially infinite collections. However, if we think about it, the whole idea seems a little bit strange: Why should the validity of a *logical* principle depend on whether the universe of discourse is finite or infinite?[1]

[1]The character named CLASS in Heyting (1971) makes the point succinctly thus: "I never understood why logic should be reliable everywhere else, but not in mathematics".

But, if we look closer, finiteness does not seem to play an *essential* role in the intuitionistic critique. In fact, the reason given by Brouwer, and later by Heyting (1971, p. 1–3), for why the universal validity of the excluded middle should be rejected involves the observation that this law embodies an unjustified metaphysical principle: the principle that mathematical objects exist independently of our knowledge of them, i. e., independently of being constructed. Thus, what the intuitionist really claims is that we should not *assume* the classical metaphysics of mathematical objects and, further, that only on the basis of this additional assumption the law of excluded middle can be justified.

Indeed, if the reasoning above is sound and we insist that logical principles should be universally applicable, we are forced to accept the conclusion that the principle of excluded middle is not a purely logical principle. Therefore, it seems reasonable to interpret the intuitionistic objection as a general objection and a contribution not only to the debate in the philosophy of mathematics but also to the philosophy of logic.

Futhermore, understood in its full generality, the contrast drawn by the intuitionist between mathematics and other subject matters dealing with finite collections also suggest that the contention lies rather in the assumption of general solvability of mathematical problems. That is, whether we are talking about finite collections of a concrete nature or about platonic mathematical objects, we can assume that for every problem or question there is a definite answer, no matter if we can find it or not.

In fact, Brouwer's method of weak conterexamples rests on the idea that, from a constructive perspective, classical reasoning would yield correct results only under the assumption of general solvability of all mathematical problems. By pointing out examples of the application of the principle of excluded middle which leads, under an intuitionistic interpretation, to the conclusion that we should possess a proof for a hitherto unproved conjecture, he attempted to show the invalidity of some classical principles of reasoning. Brouwer (1908, p. 3) realized that these principles could not be shown invalid by familiar classical counterexamples:

> In infinite systems the principium tertii exclusi is as yet not reliable. Still we shall never, by an unjustified application of the principle, come up against a contradiction and *thereby* discover that our reasonings were badly founded. For then it would be contradictory that an imbedding were performed, and at the same time it would be contradictory that it were contradictory, and this is prohibited by the principium contradictionis.

To better illustrate Brouwer's method of weak conterexamples, we consider some particular mathematical conjecture. Thus, let A express Goldbach's conjecture. From an intuitionistic point of view, a mathematical statement expresses the realization of a mental construction. Consequently, $A \vee \neg A$ means that we either have a proof or a refutation of Goldbach's conjecture. Unfortunately, we can not claim, at the moment, to have either one. Therefore, the principle of excluded middle is not intuitionistically valid.

Certainly, the intuitionist has a strong case. By dint of his rejection of the classical metaphysical view about mathematics, he was able to throw doubt on a logical principle by arguing that it is unjustifiable without appeal to classical metaphysics. But, to oppose the metaphysical doctrine of the classical mathematician, which postulates a realm of platonic mathematical objects, the intuitionist advances a metaphysical doctrine of his own: that of a mathematical reality composed solely of mental constructions.

As it is often the case with metaphysics, especially when stated so figuratively as we have been doing, the intuitionistic idea of mathematical construction is in desperate need of more careful and detailed explanation. Not by chance, some authors proposed different scenarios of how mathematics should develop under intuitionistic doctrine. One interesting example is the negationless mathematics of Griss (1946). Another important issue is whether an intuitionist should accept as justification constructions which are possible to effect in princible but were not carried out, and, maybe can not be actually carried out. For instance, let B express the claim that

(1) The sequence of digits "49027365293754" occurs in the decimal expansion of π somewhere before the $10^{10^{10}}$ decimal place.

Can we, from an intuitionistic point of view, correctly assert $B \vee \neg B$? Some people might say that to tie mathematics up with completely effected constructions made by actual persons, even the entire human race (past, present and future), is to make mathematics depend too much on casual contingent facts. According to them, a general method pertaining to (1), say using Archimedes method of calculating the decimal expansion of π, justifies the assertion of $B \vee \neg B$. To avoid losing ourselves into terminology and exegesis, let us just assume that the position we described is the accepted intuitionistic position. But, then, why stop at intuitionism? Why not embrace some sort of finitism? After all, finitism seems to agree perfectly with the general constructive position that mathematical statements express the realization of a mental construction.

These questions were not meant as rhetorical, although we shall not attempt to answer them. Actually, these are important philosophical questions concerning the objectivity and nature of mathematics. But, we used them here only with the purpose of calling attention to the fact that intuitionism is mainly a metaphysical doctrine in the philosophy of mathematics and, despite very reasonable claims to its bearing in the philosophy of logic, its contributions in the field of logic are not easy to sort out.

2.1.2 The BHK interpretation

As a disciple of Brouwer, Heyting (1930) tried to codify the principles of reasoning acceptable to the intuitionist mathematician. His first formulation was an axiomatic system in the Hilbert style. There were also some attempts by the russian mathematician Kolmogoroff (1932) to interpret the intuitionist understanding of the logical constants in terms of solutions to problems. Finally, Heyting (1971, p. 102) gave a definitive formulation known as the BHK interpretation of the logical constants. Below we have Heyting's formulation of the intuitionistic meaning of the propositional connectives[2].

$p \wedge q$ can be asserted if and only if both p and q can be asserted.

$p \vee q$ can be asserted if and only if at least one of the propositions p and q can be asserted.

$p \supset q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q.

$\neg p$ can be asserted if and only if we possess a construction which from the supposition that a construction that proves p were carried out, leads to a contradiction.

Sometimes, Heyting's clauses are adapted. Troelstra and van Dalen (1988, p. 9), for example, define negation as in Section 1.4 and add a clause to the effect that there is no proof of $\bot$. They also replace Heyting's notion of *assertion* by a

[2]Heyting (1971, p. 103) insists that the clauses apply to actual propositions and he uses germanic letters to distingish between propositions and propositional variables. In this context, generality is achieved by an additional clause: "A logical formula with propositional variables, say $\mathfrak{A}(p,q,\ldots)$, can be asserted, if and only if $\mathfrak{A}(\mathfrak{p},\mathfrak{q},\ldots)$ can be asserted for arbitrary propositions $\mathfrak{p},\mathfrak{q},\ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{A}(\mathfrak{p},\mathfrak{q},\ldots)$". We maintain Heyting's choice of letters so as to call attention to the fact that the clauses pertain to actual sentences (or propositions). The intuitionistic position on this matter is later embraced by Dummett (1991), as we see in Section 3.1.

notion of *proof*, thus tacitly assuming that we can correctly assert a proposition when we have a proof of it. Using their adaptation and some intuitive notion of proof we can argue for the validity of some logical laws.

$A \supset \neg\neg A$ There is no proof of $\bot$. So, once in possesion of a proof of A, it is impossible to have a proof of $\neg A$, that is of $A \supset \bot$, since that would in fact yield a proof of $\bot$.

Notwithstanding its initial plausibility as an explanation of the meaning attributed to the logical constants by intuitionists, the BHK clauses face many problems as satisfactory semantic clauses for a systematic theory of meaning. For certain, in mathematical contexts, is very hard to deny that we are only entitled to assert a sentence A when we have a proof of A. Still, proof can not be all there is to the meaning of mathematical statements. Otherwise, what is the meaning of a mathematical conjecture?[3]

The BHK interpretation, especially the clause for implication, also suffers from some technical problems which we mention in Chapter 3. From the perspective of developing an adequate semantics which is faithful to the ideia that meaning should be based on use, we see no reason to give any special status to the BHK interpretation. However, as we discussed at the begining of this chapter, proof-theoretic semantics has also been associated with constructive logic and mathematics. Thus, in order to also achieve the objective of providing a semantics for justifiying intutionistic logic, proof-theoretic conceptions of validity based on introduction rules (as seen in Section 3.2) have been developed under the shadow of the BHK interpretation. As Dummett (2000, p. 269) says:

> There is no doubt, however, that the standard intuitive explanations of the logical constants [BHK] determine their intended intuitionistic meanings, so that anything which can be accepted as the correct semantics for intuitionistic logic must be shown either to incorporate them or, at least, to yield them under suitable supplementary assumptions.

Admittedly, if we compare the BHK clauses above with the rules of natural deduction in Chapter 1, especially the introduction rules, we see that they are mostly similar. The similarity, however, is deceptive. The differences are significant enough to advise that natural deduction and BHK be kept safely apart. We

[3]Some mathematicians might say that the wisest thing to do with a conjecture is to remain silent about it until we have something relevant to say, that is, until we have a proof. It seems, then, that mathematical practice has not always been on the wisest path.

list bellow two noticeable differences between the BHK interpretation and natural deduction.

- the BHK clauses are formulated in terms of *proofs* (assertions) while natural deduction rules expresses the conditions to infer a sentence based on *assumptions*, i. e., hypotheses
- the BHK clause for implication is substantially different from $\supset$I[4]

Surely, the first point above is a source of inspiration for the substitutional view of open proofs. There are some problems with this view, some of which we already mentioned. As we remarked a moment ago, one of the problems is how to account for the meaning of conjectures. In contrast, an approach based on deductions from assumptions does not face the same problem: conjectures, whilst not established, can still be used in our deductions as premisses in order to extract consequences.

Yet another problem relates to the meaning of $\perp$: Granted that meaning is defined in terms of proof, or conditional proof, what is the meaning of this constant that, by definition, has no proof? Faced with this problem, Griss (1946) abandoned negation altogether. But, as we remarked on Section 1.4, $\perp$ is only expected to be used in subordinate deductions. In this regard, the approach from assumptions sugested by natural deduction also seems to provide a way out of the dilemma. Inasmuch as the motivation behind them was distinct, it is not surprising that there would be differences between natural deduction and BHK.[5]

> My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return. In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a "calculus of natural deduction". (GENTZEN, 1935, p. 176)

[4]Prawitz (1971, Section 2.1.1) also notes that the meaning of $\supset$I is more strict than that of the corresponding BHK clause and concludes: "There is thus not a complete agreement but a close correspondence between the constructive meaning of the constants and the introduction rules".

[5]Plato (2012, § 5) investigates the origin of Gentzen's formulation of natural deduction from a historical and conceptual perspective. He explores the possibility that Gentzen took inspiration not from the BHK interpretation directly but from the axiomatic systems of Hilbert, Bernays and Heyting. According to him, these axiomatic systems were coupled with Gentzen's intuition that "actual mathematical reasoning proceeds by hypotheses or assumptions, rather than instances of axioms."

Apart from its strong BHK orientation, one aspect of current proof-theoretic proposals that was directly inspired by natural deduction is the notion of *canonicity*. This notion has its inspiration on harmony and normalization procedures. However, it seems that it was brought into the scenerio only to salvage the BHK clauses from the imprecise and impredicative character of its formulation. No doubt, proof-theoretic semantics faces the challenge to sort out the conceptual role and the respective contributions of the BHK interpretation and of natural deduction in a satisfactory semantics.

2.2 Meaning and use

The idea that meaning relates to use is indeed a simple one but its correct understanding requires some explanation. First, a common misconception is that we are somehow to equate the meaning of an expression with its use, any use. Such a simple-minded approach faces two problems: first, it is not able to sistematicaly distinguish between correct and incorrect use; and, second, it does not distinguish between essential (or canonical) and inessential uses, such that the meanings of expressions are not stable.

Instead, a more adequate approach is to associate meaning with a *general formulation* of the rules governing our use of the expressions of the language. For languages in general, the formulation of these rules may indeed seem to be a very daunting endeavor since, for the most part, they can not simply be extracted from explicit patterns of usage. Nonetheless, our whole linguistic practice and our belief that there are some objective standards for the correct use of expressions suggests that their use is governed by general rules which are implicit in our capacity to use the language and which constitutes the meaning of these expressions.

Despite the fact that we can reasonably recognize correct and incorrect uses of them, it is not clear what are the general rules governing the use of most common expressions in natural languages. On the other hand, the question seems to be more approachable when we are dealing solely with logical constants. Because, in the context of logical constants, it is plausible to assume two general semantic properties: compositionality and harmony.

Compositionality is indispensable in a systematic and general account of meaning because it allows us to explain the use of expressions in any sentence recursively by reference to its components in that sentence. In the case of logical constants, the recursive specification is usually made in terms of some central semantic notion like *truth* or *assertability*. As for harmony, it is certainly the most reasonable of requirements to be imposed on the general rules governing the use of an expression if the expression has any claim at all to be considered a legitimate *logical* constant (see Section 1.6).

In the next sections, we shall discuss some philosophical arguments and theses about language and language use that are relevant for proof-theoretic semantics. In particular, we examine Dummett's notion of assertability as a satisfactory basis for the theory of meaning and his arguments for abandoning theories of meaning built around the classical notion of thruth. We also discuss the effect that some of Dummett's philosophical positions have on his own proof-theoretic semantics.

2.2.1 Manifestability

In the Introduction, we mentioned that semantical theories can be classified according to their subjacent theories of meaning. Thus, denotational theories of meaning provide a basis for model-theoretic semantics while theories of meaning based on use provides a basis for proof-theoretics semantics. In general, theories of meaning are concerned with the following question:

(2) What is the meaning of a sentence?

Framed in this general way, the question is somewhat vague. One may be asking, for instance, "how does a sentence (as a sequence of symbols) get its meaning?". This particular way of reading (2) makes it a very broad question in the philosophy of language: it demands a general theory of meaning for the language in question. Such a general theory may not produce a direct answer to particular instances like "What is the meaning of '2+2=4'?". However, it does need to provide a general framework on which this questions may be answered.

If we restrict ourselves to logical and mathematical statements, the main question for a theory of meaning is

(3) What is the meaning of a mathematical (logical) statement?

There are, at least, two answers. First, we can think of a general framework for explaining the meaning of a mathematical statement along the lines set out by Frege and other logicists. On this framework, a mathematical or logical statement is a description of certain immaterial and atemporal objects. These objects have a reality all of themselves, independent of human cognition, practice and even existence. This is the realist answer.

Another approach, which is in clear opposition to the first one, holds that the meaning of a mathematical or logical statement has to be explained in terms of its use. And, although they can be used in many different contexts and situations, the primary use of mathematical and logical statements is in mathematical calculations and proofs. This is the anti-realist answer.

Regarding a theory of meaning for logical and mathematical statements, Dummett (1978, p. 216–218) has put foward two reasons for siding with the anti-realist. First, he shows that, from the point of view of the philosophy of language, the realist's position allows for the possibility of a widespread and, worst of all, undetectable communicative faillure. Second, he points out that the anti-realist's position is more suited to the actual practice of teaching the language and can give a clearer account of this practice.

Let us take a closer look at those two arguments. The argument for the possibility of generalized communicative faillure under the realist account of meaning runs roughly as follows. Suppose that the *meaning* of a mathematical statement is indeed given by the mathematical reality it describes. Granted that this mathematical reality is objective and immaterial (not given to the senses), our grasp of the meaning of a particular mathematical statement should consist in some private mental content, possibly obtained by intellectual intuition (or something of the sort). Now, the problem is how can we be sure that we associate the same content, that is, the same meaning, to the same sentence? In other words, how can a speaker of the language be sure that he associates to a given mathematical term, say "2^3", the same meaning that some other speaker? The answer is simple: if all there is to the *meaning* of a mathematical statement is a correspondence to an immaterial mathematical reality, one speaker can never be sure to understand such a statement the same way another speaker does.[6]

For very much the same reasons above, a teacher of the language cannot be sure to have teached it correctly. The learner also cannot be sure to have learned it correctly. However, it is a well-established fact that we do teach and learn mathematics. And we do it by observing and correcting each other when using mathematical expressions. Thus, any model of meaning completely alien to use cannot give a satisfactory account of the teaching and learning of mathematics.

All the discussion so far is well-known and has been retold many times in the literature. However, it raises more questions than it answers. For instance, what does it mean for the realist to hold that there are immaterial and atempotal

[6]This argument is not exactly the same one given by Dummett (1978, p. 216–218). There, Dummett pressuposes the thesis that the meaning of a mathematical statement is determined by its use and proceeds from there. In contrast, we gave a kind of *reductio ad absurdum* of the realist position.

mathematical objects inhabiting a reality independent of ourselves? What is the effect of the appeal to use as a fundamental element of meaning? How does the anti-realist's theory of meaning differs from the realist's? No doubt, one of Dummett's more outstanding contributions to the philosophy of language and to metaphysics is his attempt to relocate some metaphysical disputes over to the semantic camp. According to him, the core of the dispute between the realist and the anti-realist can be rendered as a dispute around the legitimacy of the principle of bivalence when applied to a certain class of sentences, called the disputed class. For instance, in the philosophy of mathematics, where anti-realism is represented by intuitionism and other forms of constructivism, the dispute class is, obviously, the class of mathematical sentences.

The rejection of the principle of bivalence is believed to follow from an anti-realist commitment to language use. That is, an adequate theory of meaning based on use can not justify the principle of bivalence as a valid semantic principle. Thus, the appeal to use on the part of the anti-realist can be seen in the form of a general requirement placed on the theory of meaning. This requirement, called the requirement of manifestability, are designed to exclude conceptions of meaning based upon a verification transcendent notion of truth.

Manifestability In a theory of meaning, the explanation of the meaning of an expression must be made only in terms of notions and distinctions which are completely manifestable in the linguistic behavior of the community.

It is important to understand clearly what is the danger the anti-realist is trying to avoid. In other words, we need to undestand exactly what are those use transcendent meanings that can be assigned to expressions by bivalent denotational theories of meaning. Regarding the differences between the anti-realist (in this case represented by the intuitionist position) and the realist theories of meaning, Dummett (1975b, p. 22) states:

> There is no substantial disagreement between the two models of meaning so long as we are dealing only with decidable statements: the crucial divergence occurs when we consider ones which are not effectively decidable.

What do the anti-realist have to say about undecidable statements and their meaning? Are they meaningless? According to Dummett (1991, p. 315), some features of our language allows for the construction of undecidable statements out of ordinary expressions whose meaning can be fully explained by reference to their use in decidable ones. These features are:

- Our capacity to refer to inaccessible regions of space-time, such as the past and the spatially remote.
- The use of unbounded quantification over infinite totalities.
- Our use of the subjunctive conditional.

Therefore, the manifestability requirement does not deprive undecidable statements of meaning, since they are natural linguistic constructs made possible by some general devices of our language. However, it does deprive undecidable statements of the privilege of bivalent truth.

2.2.2 Assertion

Among the speech acts we use, assertions are the ones most relevant when it comes to knowledge in general and logic in particular. In verificationist theories of meaning, assertability is brought to the foreground. Thus, instead of the concept of truth, familiar from classical theories of meaning, a theory of meaning committed to language use will take assertability as its central semantic concept. When he discusses the uses we make of assertions, Dummett (1991, p. 103) often mentions the aspects we already treated in connection with the deductive use of the logical constants:

> There are two aspects of the use of any assertoric sentence, which provide the answers to the questions, "When should I use it?" and "What can I do with it?" To know when I should use the sentence is to know what evidence establishes it as true and from what premisses it may be inferred. To know what to do with it is to know what bearing its truth may have on my actions; and this involves knowing what consequences flow from it, together with other statements accepted as true, and how such consequences may affect the outcome of my actions.

Moreover, in contrast with the abstract concept of *proposition*, assertions, as speech acts, carry with them an implicit commitment, by the person making the assertion, to stand for its correctness. In other words, the person making the assertion can be challenged to offer justifications for it and thus to make explicit the grounds on which rest his knowledge of its correctness. Clearly, then, the conditions to correctly assert a proposition and the conditions for it to be true (in a classical sense) are different.

The distinction between truth conditions and assertability conditions is important for the understanding of the differences between classical and verificationist theories of meaning. For instance, Martin-Löf (1996, p. 23) formulates the distinction thus:

> Even if every even number is the sum of two prime numbers, it is wrong of me to say that unless I know it, that is, unless I have proved it. [...] So the condition for it to be right of me to affirm a proposition *A*, that is, to say that *A* is true, is not that *A* *is* true, but that I *know* that *A* is true.

A problem we discussed in Section 2.1.2 reappears here: if the criterion for an assertion to be correct is for us to have a justification for it, and, if meaning are to be explained in terms of assertability, then what becomes of conjectures or, for that matter, of suppositions "for the sake of argument"? The other aspect mentioned by Dummett in the quote above, "What can I do with it?", may account for them. But, then, it conflicts with the idea that the correct assertion of a sentence demands conclusive proof or evidence from the person making the assertion.

A way to conciliate our intuitions is to introduce the notion of *conditional assertion*. Thus, we can say that the assertion of some particular sentence *A* is conditional on the assertion of *B* and *C*, for example. In other words, in a subordinate argument, we are conditionally asserting the conclusion, that is, we claim that it would be correct to assert *A* on the condition that it is correct to assert *B* and *C*. In full generality, the idea implies that every assertoric sentence of the language should be provided with assertability conditions. As a result, we can see that the notion of *assertion* influenced considerably some current proof-theoretic accounts of the semantics of logic. The influence can be felt, for instance, in the notion of *basic rules* B which provide assertability conditions for atomic sentences. These rules license the assertion of atomic sentences from atomic sentences.

Consequently, most proof-theoretic accounts of logic figure a notion of valid canonical argument as trees proceeding from atomic premisses, or atomic derivations by basic rules, such that earch node represents an assertion, or conditional assertion, and each step is a passage from assertions to assertions. Certainly, this approach has the generality needed if we are trying to give a complete verificationist account of the whole language (purely descriptive sentences included). In an account of logical validity, however, it has the cumbersome effect that, in principle, we have to deal with bases and basic rules. Of course, the assertability conditions of any *particular* descriptive atomic sentence have little, if anything, to do with logical validity[7]. Notwithstanding, proof-theoretic definitions of valid-

[7] Sure enough, we understand that, regarding basic systems, the main concern is not purely de-

ity, like that of Prawitz (1971), end up with formulations relative to any basis B (or arbitrary monotonic extensions of them). Nonetheless, even with the help of the concept of *conditional assertion* some difficulties persist. There remains the problem of the subordinate arguments for $\bot$: we are not willing to assert $\bot$ under any condition whatsoever.

If we abandon the notion of valid canonical argument as described above and, instead try to explain validity on the basis of correct inferrability from assumptions along the lines sketched on Chapter 1, we may be able to avoid the need for bases and basic rules altogether.

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scriptive sentences but some especial non-logical, yet non-descriptive, mathematical sentences. As mentioned earlier, we are interested exclusively in logic and shall not consider foundational mathematical questions directly (save for its historical influence in proof-theoretic semantics for logic). Nonetheless, as a side note, I (the author) believe that the unwavering temptation to accommodate an account of the necessity and objectivity of mathematical sentences has been responsible for part of the problems faced by most theories of logic since the good old times of Frege and Russell.

Chapter 3

Proof-Theoretic Validity

The standard constructive explanation of the logical constants, the so called BHK interpretation, doesn't lend itself easily to the inductive treatment required for a definition of validity. The BHK clause for implication, for instance, seems to be strongly impredicative. It refers to *any* construction of the antecedent and that might involve a construction of the implication itself (for example, in a "roundabout" proof). This problem is pointed out by many authors, among which are Gödel (1995) and Dummett (2000, Section 7.2). Díez (2000, p. 410), for instance, formulates the problem as follows:

> The construction which is being defined and which proves $p \supset q$ must be able to transform any possible proof of p into a proof of q; as no boundary is put on the complexity of those possible proofs of p, they could include some complicated roundabout proofs which involved reference to the sentence $p \supset q$ itself, and hence to the same proof being defined. In sum: the definition of a proof of $p \supset q$ appeals to a totality of proofs, with some of which the very proof of $p \supset q$ could be intimately related.

Nevertheless, relying on the notion of *canonicity* made viable by the harmonious behavior of natural deduction rules, Prawitz (1971) and Dummett (1991) have proposed proof-theoretic inductive definitions of validity intended for justifying predicate intuitionistic logic. Yet, there has been some debate about whether these definitions are correct. Sandqvist (2009), for instance, have stated that some proof-theoretic approaches would in fact yield a constructive justification of classical logic. Sanz, Piecha and Schroeder-Heister (2012), in particular, argued that Prawitz's proposal results in a conflation of admissibility[1] and derivability for the

[1] A rule r is *admissible* in a formal system S if, for every sentence A, whenever there is a closed

fragment $\{\supset\}$ of propositional logic. They have shown that Peirce's rule for basic sentences is admissible and thus valid according to Prawitz's definition. As we saw on Chapter 2, a proof-theoretic justification of classical logic will make a strong impact on Dummett's program.

The purpose of this chapter is to evaluate the applicability of those same criticisms to Dummett's proposal. In what follows, we shall review Dummett's definition of validity in detail, giving references whenever necessary to support the formulations[2]. Sometimes, we simplify references by adopting the convention that unqualified page references in this chapter always refer to Dummett (1991). But, before stating the definitions, we explain briefly some peculiarities of Dummett's approach.

Dummett (1991) considers first, second and third grade proof-theoretic justifications for logical laws. Each one is a more powerful justification procedure than the previous. Justifications of the first grade consists merely in the derivation of a rule from other ones taken as given. In other words, for this kind of justification we assume that some given set of rules are valid and try to justify other rules by *deriving* then from the given set.

On the other hand, second grade justifications introduce the main idea of proof-theoretic semantics: the introduction (or elimination) rules fix the meaning of the logical constants they govern. As we saw on Section 1.6, this insight dates back to some remarks of Gentzen to the effect that the introduction rules are definitions of which the elimination rules are only "consequences". From the verificationist point of view adopted by Dummett (1991, p. 246), the legitimacy of this idea rests on the fact that, if the introduction and elimination rules for a given constant are in harmony (the elimination rules are only "consequences" of the introduction rules or vice-versa), then the addition of these rules yields a conservative extension of the language.

In order to give a more precise content to Gentzen's remarks, Dummett develops a general justification procedure which, given a set I of introduction rules, would validate all other rules with respect to them, including the corresponding elimination rules for the relevant logical constants. The justification procedure amounts to a definition of validity *with respect to a given set* I *of introduction rules*. We do not follow Dummett's terminology when we call it the *verificationist justification procedure*. Another procedure, the *pragmatist justification procedure*, validates inference rules with respect to a given set E of elimination rules.

proof of A in the system S extended by the rule r, then there is a closed proof of A in the system S without the rule r (assuming that r did not initially belonged to S).

[2]We avoid explicit quotations so as to keep the chapter within reasonable proportions. Should any doubt arise, the reader is advised to look up for himself the relevant passages.

Third grade proof-theoretic justifications are introduced to deal with discharged hypotheses (p. 259–260) and certain kinds of ancillary premisses (p. 282–283), in the case of introduction and elimination rules, respectively. Dummett (1991, p. 286) claims that both justification procedures assures us of *intrinsic harmony*, that is, harmony between the introduction and elimination rules for a given constant γ, as discussed in Section 1.6.

However, there is still a further requirement: the *requirement of stability*. This requirement appeal to both kinds of justification procedures as means to guarantee that the set I [E] of rules collectively determine a coherent inferential practice (p. 287–289). Dummett believes that, in this case, we achieve not only intrinsic harmony, but also *total harmony*. So, to establish total harmony, the *requirement of stability* needs to be satisfied.

3.1 Preliminary notions and definitions

The following definitions will be used in our discussion of both the verificationist and the pragmatist justification procedures. They deal with the notions of argument and rule of inference. As a matter of fact, an intuitive understanding of these notions have been assumed in our discussion so far. The definitions are not meant to replace the intuitions but rather to avoid misunderstandings with the techinical material in next sections.

An *argument* Π is a tree composed of sentences. The sentence at the root of the tree is called the *conclusion* of the argument. If an argument Π is composed according to a system of rules S, we say that Π is a *deduction* in S. We understand arguments as being composed from top to bottom, that is, from leaves to root. Every sentence A occuring in a path from a leaf to the root of an argument Π determine a subargument Π_1 in the obvious way with A as its conclusion. A top occurence of a sentence A can be *discharged* by the application of a rule r yielding a conclusion B which, together with all the other sentences below, do not depend on A. Informally, we tend to use “hypotheses” for top sentences that were discharged somewhere in the argument under consideration and “premisses” for sentences that were not discharged.

It is important to notice that, following Heyting (1971, p. 101), Dummett (1991, p. 255–256) considers deductive arguments to be composed of actual sentences and thus his definitions apply to these concrete arguments. Accordingly, to show the validity of a schema representing an inference rule we need to establish the validity of any application of it, that is, of any argument resulting from the substitution of actual sentences for the schematic letters. However, because propositional logic should be sufficient for our purposes, the use of schematic letters to stand for actual sentences does not pose any special problems[3]. For the same reason, we ommitt complications and clauses in the definitions dealing with quantification and open sentences.

Definition 1. A *sequent* is an ordered pair $\Gamma \to A$ in which Γ, called the *antecedent*, is a set of sentences and A, called the *sucedent*, is a single sentence (p. 185).

A sequent $\Gamma \to A$ represents a deduction from premisses Γ to conclusion A. They are not considered to be among the sentences appearing in argument trees but are only a device intended to simplify our notation by spelling out what are the premisses on which depends some occurence of a sentence in an argument tree (p. 254).

Definition 2. An one-step *argument* is an ordered pair $\langle \Gamma_1 \to A_1 \ldots \Gamma_n \to A_n, \Delta \to B \rangle$ consisting of a set of sequents $\Gamma_1 \to A_1 \ldots \Gamma_n \to A_n$, called the *base sequents*, and a single sequent $\Delta \to B$, called the *resultant sequent*.

An *inference rule* is represented by an one-step argument schema. The definition of an one-step argument, which takes arguments to be a transition from sequents to sequents is needed mainly in the general case, for instance, when we are dealing with rules that discharge hypotheses (p. 186 and 264). In such general cases, we may talk about the sucedent B of the resultant sequent simply as the *conclusion* and its antecedent Δ simply as the set of *premisses*. We call a *discharged hypothesis* any sentence A such that A is in the union $\Gamma_1 \cup \cdots \cup \Gamma_n$, but is not in Δ. Otherwise, when we are dealing with rules of a more simple kind, they can be represented simply as a single sequent $\Delta \to B$ where the sentences in Δ are the premisses and B is the conclusion. This distinction between rules is also employed by Prawitz (1965, p. 22–23), where he distinguishes between "inference rules" and "deduction rules".

We use the familiar natural deduction representations of arguments and argument schemas and do not bother with spelling out the premisses of each sentence occurence by using sequents. Since we shall not discuss any complex inference rule discharging hypotheses, the usual natural deduction representation is sufficient and more clear. For instance, Peirce's rule may be represented with $(A \supset B) \supset A$ as premiss and A as conclusion as shown below.

[3] Since we don't need to worry about open sentences, we just let the schematic letters stand for an arbitrary closed sentences. In this way, our reasoning is guaranteed to yield a general method applicable to any actual deduction.

$$\frac{(A \supset B) \supset A}{A}$$

According to the standard proof-theoretic view, for a complete semantic specification, besides inference rules governing the assertability of complex sentences, we also need criteria for the assertability of atomic sentences. Usually, these criteria are given by a set of rules B, called *boundary rules* or *basic rules*, of the form $\alpha_1 \ldots \alpha_{n-1}/\alpha_n$ allowing the inference of atomic sentences from other atomic sentences. In fact, the exact form of the basic rules are controversial. Prawitz (1971, p. 276) takes them to be production rules that don't discharge hypotheses and whose conclusion are always atomic. Dummett (1991, p. 186), on the other hand, allows for complex conclusions in boundary rules, however his definition of validity with respect to a given set I [E] of introduction [elimination] rules are such that they assume the basic rules to be production rules with atomic conclusions, as can be seen from the definitions given below[4].

3.2 Verificationism

In their paper, Sanz, Piecha and Schroeder-Heister (2012) prove, in the fragment $\{\supset\}$ of natural deduction, that given a proof of the premiss of Peirce's rule for atomic φ and ψ, we have a proof of the conclusion. Because of the way Prawitz (1971) frames his inductive clause for $\supset$, this amounts to a validation of Peirce's rule in his semantics. Adapting their argument to Dummett's approach, we need to show whether Peirce's rule is valid or invalid given any basis B of production rules (or, as Dummett calls them, boundary rules) and the set I whose sole rule is $\supset$I. Since $\supset$I may discharge hypotheses, we must use third grade proof-theoretic justifications. That will be the case for the remainder of the text. Now, before proceeding to the verificationist definition of validity, we need to fix some terminology.

Definition 3. A sentence occurrence is in the *main stem* of an argument Π if every sentence intervening between it and the conclusion of Π (inclusive) depend only on the premisses of Π. Otherwise, if a sentence occurrence that is not in the main stem lies immediately above one that does, then it is the final conclusion of a *critical subargument* (p. 260).

As we can easily see, the purpose of the concept of *main stem* is to keep track of the discharge of hypotheses as we move up from the root of the argument to-

[4]According to Definition 3, a canonical argument with complex conclusion is required to have as its last step an application of an introduction rule.

wards its leaves. Meanwhile, as we thus move, examining each of the possible branches in turn, we identify subarguments whose conclusion depend on additional premisses. We recall that, with regard to the discharge of hypotheses, as we remarked on Section 1.1, the rule $\supset$I specifies two distinct conditions for its application. Thus, when faced with an application of $\supset$I to conclusion $A \supset B$, the occurence of B immediately above $A \supset B$ is, or is not, in the main stem depending on whether an hypotheses was actually discharged.

Example 1. In the following example, only the conclusion $(B \wedge C) \supset (A \wedge B)$ is in the main stem. All the deduction up to that point is a critical subargument.

$$\dfrac{\dfrac{A \qquad \dfrac{[B \wedge C]}{B}}{A \wedge B}}{(B \wedge C) \supset (A \wedge B)}$$

It is interesting to notice that, when considering the example above, Dummett (1991, p. 263) doesn't follow his own definition. He claims that both the premiss A and the final conclusion $(B \wedge C) \supset (A \wedge B)$ are in the main stem. However, since the sentence $A \wedge B$ (which depends on the hypotheses $B \wedge C$) occurs in the path from A to the conclusion, we must admit that A is not, after all, in the main stem. Still, this blunder does not affect the validity of the inference of $(B \wedge C) \supset (A \wedge B)$ from the premiss A for which he was arguing in this passage.

Definition 4. We say that a given argument is *canonical*, valid or not, if the following conditions hold:

(i) all its (undischarged) premisses are atomic sentences;

(ii) every atomic sentence in the main stem is either a premiss or is derived one of the basic rules;

(iii) every complex sentence in the main stem is derived by means of one of the introduction rules.

The notion of a canonical argument, as expressed by the definition above, features some important properties. As a consequence of item (iii), we can be sure that canonical arguments always have an application of an introduction rule as its last step (assuming the conclusion to be a complex sentence). This property is also emphasized by Prawitz (1971, 2006). From examination of item (ii), we also notice that canonical arguments whose conclusion are atomic sentences proceed only by basic rules from atomic premisses (if there are any) since the atomic conclusion is, by Definition 3, on the main stem. With respect to item (i), however, Dummett deviates from other authors, in particular from Prawitz (1971), when he accepts atomic premisses to stand undischarged (open) in canonical arguments.

Definition 5. A *supplementation* of a given arbitrary argument is the result of replacing each complex premiss by a valid canonical argument having that premiss as its conclusion (p. 261).

As we shall notice latter, in Definition 6, transformations of supplementations are the main element in the verificationist justification procedure. The notion of a supplementation will become completely clear only after we define valid canonical arguments. We can clearly see Dummett's unwavering adhrence to the substitional view of open proofs when he defines supplementations as the result of substitution of valid canonical argument for complex premisses. Another important thing to notice is that the premisses of a supplementation are exactly those of the valid canonical arguments substituted plus any other atomic premisses already in the supplemented argument.

Example 2. When evaluating the validity of an inference rule, in order to come up with a general procedure, we need to consider every possible supplementation of its premisses. Then, by refletion on the definitions, we obtain some properties which can help us to effect the necessary transformations. Thus, let us consider the following inference rule.

$$\frac{B \vee C}{(A \supset B) \vee (A \supset C)}$$

Now, if we take our set I of introduction rules to contain $\vee$I and $\supset$I, the supplementations of any argument resulting from the substitution of actual sentences for the schematic letters of the inference rule above will take one of the forms below, where $\alpha_1 \dots \alpha_n$ and $\beta_1 \dots \beta_n$ are atomic premisses.

$$\cfrac{\cfrac{\begin{array}{c}\alpha_1 \dots \alpha_n \\ \Sigma_1 \\ B\end{array}}{B \vee C}}{(A \supset B) \vee (A \supset C)} \qquad\qquad \cfrac{\cfrac{\begin{array}{c}\beta_1 \dots \beta_n \\ \Sigma_2 \\ C\end{array}}{B \vee C}}{(A \supset B) \vee (A \supset C)}$$

We also note that if B and C are atomic then, by item (ii) of Definition 4, both subarguments determined by Σ_1/B and Σ_2/C are derivations from atomic premisses $\alpha_1 \dots \alpha_n$ and $\beta_1 \dots \beta_n$, respectively, proceeding solely by basic rules.

Definition 6. An arbitrary argument is *valid* if we can effectively transform any supplementation of it into a valid canonical argument with same premisses and conclusion. Furthermore, a canonical argument is *valid* if all its critical subarguments are valid (p. 261).

Although Dummett (1991, p. 64) demands an argument to be an actual complete deduction composed of sentences, he claims that his definition "takes no overt account of more than the initial premisses and final conclusion of the argument". In other words, despite being applicable directly to *arguments*, Definition 6 can be used to construct a general procedure capable of validating one-step *argument schema* (or inference rules) with respect to a given set I of introduction rules.

Finally, in order to the definition of validity from a verificationist perspective to be complete, there is still a major ingredient to be discussed: the basic rules. As we already saw in Section 3.1, these rules license the inference of atomic sentences from atomic sentences. With the adoption of the apropiate set of basic rules, it is possible to formalize the deductive relations that appear in mathematical theories. Much of the interest for the investigation of basic rules is because they provide a general framework for the study of formal systems, not only of logic, but also of mathematics, even though basic rules does not pertain, at least diretly, to the notion of *logical* validity.

However, Dummett's definitions for the verificationist justification procedure do incorporate basic rules and there is no way around them, even though his definitions are intended for logical validity. Throughout his book, he seems to waver between the benefits offered by basic rules on the one hand and, on the other hand, the intuition that basic rules are irrelevant in the justification of purely logical rules of inference. In fact, Dummett (1991, p. 273) explicitly asserts the independence between basic (or boundary) rules and the notion of logical validity:

> We originally admitted, as occurring within deductive proofs of the kind with which we are concerned, boundary rules allowing the inference of an atomic conclusion from atomic premisses: these were, of necessity, left unspecified. Our original intention was that the boundary rules should be deductively valid. If we now include among them principles of non-deductive (and therefore fallible) inference, this will have the effect that a 'valid' argument, even if canonical, may have true initial premisses but a false final conclusion. *It will obviously not affect the justification procedure, however, as a means of determining the validity of logical laws.*

Unfortunately, since Dummett adopts the substitutional view of open proofs, the basic rules become an indispensable element in the verificationist justification procedure. Granted that the explicit consideration of basic rules cannot be avoided, the question now is how to generalize the justification procedure such that we obtain logical validity instead of validity with respect to some particular set of basic rules. With respect to this question, Dummett (1991) himself does not offer many clues. Judging from the quotation above, he seems to assume that an arbitrary set of basic rules are somehow given and that the justification procedure can be applied without relying on any assumption about the rules or their structure. If the basic rules are indeed irrelevant for the analysis of logical validity, the right approach would be to leave them out of consideration altogether. However, the justification procedure that come out of Dummett's definition cannot possibly avoid the explicit consideration of basic rules (see Definition 4).

There are three distinct ways that we can generalize the justification procedure with respect to the basic rules so as to arrive at a notion o logical validity:

empty base one way to avoid that any particular feature of some base interfere with the justification procedure is to consider an empty base, that is, $\mathsf{B} = \emptyset$.

all basis another way would be to require that, for an inference to be logicaly valid, the justification procedure must be capable of justifying it in all the possible bases.

monotonic extensions of basis yet another way would be to adopt the strategy of Prawitz (1971) which relies on monotonic extensions of an arbitrary base. An advantage of this strategy is that it incorporates the intuition that a sentence, once proved, remains proved. The strategy achieves generality because the base B to be extended as an arbitrary base.

Let us see which one of the three ways above is more adequate. On light of the definitions, consider the following inference, where φ and ψ are atomic:

$$\frac{\varphi \supset \psi}{\psi} \tag{3.1}$$

By Definition 6, for (3.1) to be valid any supplementatio of $\varphi \supset \psi$ must be transformed in a valid canonical argument for ψ depending on the same premisses. If $\mathsf{B} = \emptyset$, every possible suplementation involves the assumption of ψ[5]. Among them, the simplest one is:

[5]In contrast with Prawitz (1971) and others, Dummett (1991) accepts open atomic premisses in canonical arguments (see Definition 4).

$$\frac{\dfrac{\psi}{\varphi \supset \psi}}{\psi}$$

By Definition 4, the assumption of ψ is a valid canonical argument for ψ. Therefore, in the empty base, we can establish the validity of (3.1). Moreover, if we choose the empty base as the right way to account for logical validity, then (3.1) would be *logicaly* valid, which is unacceptable[6]. Also, it does not make sense to require that the supplementations to be considered are those belonging to all the basis, because their common denominator woulb be the empty base and we will be faced with the same problem.

There are still two alternatives. The first of then suggets that the definition of validity, when applied to logical inferences, requires a quantification over all bases. In other words, an inference would be logically valid only if it can be validated in all bases. This approach avoids the problem with the empty basis since the inference rule used in (3.1) above is not valid in a basis containing the rule φ/ψ.

To consider all bases is equivalent to consider a lattice of bases such that the minimal element is the empty basis and the maximal element is a basis containig all possible combinations of atomic rules. The elements of the lattice are related by inclusion, i. e., extension by addition of basic rules. Therefore, validity in all bases is equivalent to validity by monotonic extensions of the empty basis.

Theorem 1. *Let φ and ψ be atomic closed sentences. Then, given any set B of production rules, Peirce's rule*

$$\frac{(\varphi \supset \psi) \supset \varphi}{\varphi}$$

is valid in B by the verificationist justification procedure with respect to $\supset$I.

Proof. We have to show that we can effectively transform any supplementation into a valid canonical argument for φ depending on the same premisses. Suppose we have a supplementation Π_1, in an extension C of B, depending on premisses $\alpha_1 \dots \alpha_n$ obtained by substitution of $(\varphi \supset \psi) \supset \varphi$ by a valid canonical argument as specified by Definition 5. Thus, the penultimate step in Π_1 is an application of $\supset$I as shown below.

$$\frac{\dfrac{\begin{matrix}\alpha_1 \dots \alpha_n \\ \Sigma_1 \\ \varphi\end{matrix}}{(\varphi \supset \psi) \supset \varphi}}{\varphi} \qquad (\Pi_1)$$

[6]This critique is due to Warren Goldfarb.

We have two possibilities to consider: either (1) the penultimate occurence of φ is in the main stem and we already have a valid canonical argument for φ from the same premisses, by item (ii) of Definition 4, or (2) φ is not in the main stem and we have a critical subargument Π_2 with φ as conclusion (by Definition 3).

In case (2), φ depends on a discharged hypotheses (again, by Definition 3). Considering that it was discharged by an application of $\supset$I whose conclusion is $(\varphi \supset \psi) \supset \varphi$, the hypotheses can only be $\varphi \supset \psi$.

$$\begin{array}{c} \varphi \supset \psi, \alpha_1 \dots \alpha_n \\ \Sigma_1 \\ \varphi \end{array} \tag{Π_2}$$

By Definition 6, the critical subargument Π_2 is a *valid* argument. From the validity of Π_2, we show how to obtain a valid canonical argument for φ in C from atomic premisses $\alpha_1 \dots \alpha_n$. Because Π_2 is valid, we have a procedure to effectively transform any supplementation of Π_3, in an extension D of C, in a valid canonical argument for φ from premisses $\alpha_1 \dots \alpha_n$,$\beta_1 \dots \beta_n$.

$$\begin{array}{c} \dfrac{\begin{array}{c}\beta_1 \dots \beta_n \\ \Sigma_2 \\ \psi\end{array}}{\varphi \supset \psi} \;,\quad \alpha_1 \dots \alpha_n \\ \Sigma_1 \\ \varphi \end{array} \tag{Π_3}$$

In particular, in the extension $\mathsf{D}^* = \mathsf{C} \cup \{\varphi/\psi\}$, we have a valid canonical argument from premisses $\alpha_1 \dots \alpha_n$ only, although in basis D^*. By Definition 4, this valid canonical argument proceeds solely by the basic rules of D^*.

$$\begin{array}{c} \alpha_1 \dots \alpha_n \\ \Sigma_4 \\ \dfrac{\varphi}{\psi} \\ \Sigma_5 \\ \varphi \end{array} \tag{Π_4}$$

We examine the valid canonical argument for φ in D^*. If the rule φ/ψ is not used, then we have in fact a valid canonical argument in the basis C from premisses $\alpha_1 \dots \alpha_n$ and our proof is complete. Else, if the rule is used, we take its first application as depicted in Π_4 above. Since the rule φ/ψ does not occur in the subargument Σ_3/φ, we obtain the required valid canonical argument for φ in C from premisses $\alpha_1 \dots \alpha_n$. $\square$

As can be seen from the validation of Peirce's rule, the verificationist definition of validity not only affords a criterion to tell valid rules from invalid ones, but actually amounts to a justificatory procedure. Apparently, however, Dummett (1991, p. 270) did not envisaged the possibility that the verificationist justification procedure would validate Peirce's rule. A few pages after the definition of validity, he remarks:

> "On a realist meaning-theory, however, the correct logic will be classical; and there will be many classically valid laws involving those logical constants that cannot be validated by appeal to the introduction rules governing them, such as those expressed by the classically valid schemata $(A \supset B) \vee (B \supset A)$, $(A \supset B) \vee A$, $((A \supset B) \supset A) \supset A$."[7]

Indeed, once Peirce's rule is valid, we can obtain all classical tautologies in the fragment $\{\supset\}$ of the language of propositional logic. If we add $\wedge$I and $\bot$I (or $\bot$E) as rules to the implicational fragment —which, since Peirce's rule is valid, behaves classicaly—, we obtain a complete set of logical constants powerful enough to account for all valid propositional classical reasonings (with the other constants being defined in terms of implication, conjuction, and $\bot$). This state of affairs certainly frustrates the expectation that proof-theoretic validity provides justification only for constructive reasonings (see Chapter 2). Does an approach to meaning based on use lead to the justification of the same classical reasonings as the classical denotational approach? What could be the reasons for this somewhat undesirable outcome?

By Theorem 1, the assumption that we have a putative valid canonical argument for $(\varphi \supset \psi) \supset \varphi$ leads necessarily to the conclusion that we have a valid canonical argument for φ from the same premisses. The critical point in the proof is when, after the first supplementation, we assume φ not to be in the main stem and consider the critical subargument from $\varphi \supset \psi$ (possibly with some atomic premisses). The assumption that φ is not in the main stem signals the discharge of hypotheses and, in most cases, indicates that φ depends on more premisses than the conclusion of the supplementation. But, even under the assumption that it is not in the main stem, we discover, after applying the definitions, that no hypotheses was effectively discharged and, consequently, our valid canonical argument for φ depends on the same atomic premisses as the conclusion of our instance of Peirce's rule. We believe that the reason for this result lies in the restriction of the supplementations to substitution of valid canonical arguments from atomic premisses.

[7] Note that the last schema in this quotation is the so called Peirce's law which can be obtained from Peirce's rule by $\supset$I. The schema mentioned by Dummett are rendered in our own notation.

If we forget the substitutional view of open proofs and concentrate on the notion of *harmony* discussed in Section 1.6, we shall come to a somewhat different understanding of how Peirce's rule should be handled: all the conditions necessary to infer $(A \supset B) \supset A$, assuming it was infered by means of $\supset$I, should license the inference of A (there is no need to suppose A and B to be atomic). The conditions to obtain $(A \supset B) \supset A$ are depicted bellow.

$$\begin{array}{c} \Gamma \\ \Sigma_1 \\ A \end{array} \qquad\qquad \begin{array}{c} A \supset B, \Gamma \\ \Sigma_2 \\ A \end{array}$$

On the left, A is in the main stem and there was no discharge. On the right, A is not on the main stem and $A \supset B$ was discharged. We can represent these conditions by the sequents $\Gamma \rightarrow A$ and $\Gamma, A \supset B \rightarrow A$, respectively. We propose that supplementations, when needed[8], consists in the conditions to apply the corresponding introduction rule.

For supplementations with A in the main stem, as on the left above, we already have the required valid canonical argument for A. For supplementations with A not in the main stem, we do have a valid argument for A (which is the conclusion of a critical subargument) but it is not garanteed to depend on the same premisses Γ. Indeed, besides Γ, the valid argument for A depends on $A \supset B$. Although the result already seems obvious, we can analyse further since we know, By Definition 6, that from all the conditions to obtain $A \supset B$ by $\supset$I (supplementations), we can obtain a valid canonical argument for A from the same premisses. So, we continue and consider the supplementations of $A \supset B$.

$$\begin{array}{c} \Gamma \\ \Sigma_3 \\ B \end{array} \qquad\qquad \begin{array}{c} A, \Gamma \\ \Sigma_4 \\ B \end{array}$$

If B is on the main stem, there was no discharge of hypotheses and B depends, at most, on the same premisses Γ. As we remarked above, by Definition 6, this valid canonical argument for B can be transformed into one for A from the same premisses Γ, or less. However, if B is not in the main stem, the premisses are not limited to Γ but include also the hypotheses A. There is no general way to dispense with this hypotheses and, consequently, no way to obtain a valid canonical argument for A from the same hypotheses Γ. Therefore, Peirce's rule is not valid.

[8]Optionally, we can just assume complex sentences as premisses without supplementation.

The notion validity used above does not make validity ammount to a transformation of valid canonical arguments into valid canonical arguments. On the other hand, Dummett's verificationist definition of validity —based on the idea of showing that if we have a proof (valid caninical argument) for the premisses, then we have a proof (valid caninical argument) for the conclusion— resemble the notion of admissibility. It seems that the verificationist justification procedure, in the presence of $\supset$I, would validate any admissible rule involving this logical constant[9]. As a means to increase the plausibility of this conjecture, let us try to apply the verificationist justification procedure to a well-known admissible rule due to Mints (1976).

Theorem 2. *Let* φ*,* ψ *and* χ *be atomic sentences. Then, given any set* B *of production rules, Mints' rule*

$$\frac{(\varphi \supset \psi) \supset (\varphi \vee \chi)}{((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)}$$

is valid with respect to $\supset$*I and* $\vee$*I.*

Proof. We show that any supplementation will give us a valid canonical argument from the same premisses for the conclusion. Suppose a supplementation Π_1, in an extension C of B, as shown below.

$$\cfrac{\cfrac{\begin{matrix}\alpha_1 \ldots \alpha_n \\ \Sigma_1 \\ \varphi \vee \chi\end{matrix}}{(\varphi \supset \psi) \supset (\varphi \vee \chi)}}{((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)} \qquad (\Pi_1)$$

There are two possibilities. Either (1) $\varphi \vee \chi$ is in the main stem and we have valid canonical arguments for either φ or χ from the same premisses. With any one of those, we can easily obtain a valid canonical argument for $((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)$ by means of $\supset$I and $\vee$I. Or (2) $\varphi \vee \chi$ is not in the main stem, in which case the critical subargument Π_2 below is valid.

$$\begin{matrix}\varphi \supset \psi, \alpha_1 \ldots \alpha_n \\ \Sigma_1 \\ \varphi \vee \chi\end{matrix} \qquad (\Pi_2)$$

By Definition 6, any supplementation Π_3, in an extension D of C, of the critical subargument Π_2 can be effectively transformed into a valid canonical argument

[9] Also, this is basically what is argued by Sanz, Piecha and Schroeder-Heister (2012) for the semantics of Prawitz (1971).

for $\varphi \vee \chi$, no new premisses required besides those eventually introduced by the supplementation procedure. Similarly to the proof of Theorem 1, we consider the extension $\mathsf{D}^* = \mathsf{C} \cup \{\varphi/\psi\}$. In $\mathsf{D}^* = \mathsf{C} \cup \{\varphi/\psi\}$, we have a valid canonical argument for $\varphi \vee \chi$ from premisses $\alpha_1 \ldots \alpha_n$. Next, still following the proof of Theorem 1, we obtain a valid canonical argument for φ from $\alpha_1 \ldots \alpha_n$, now in the basis C. Finally, by application of $\supset$I and $\vee$I, we construct a valid canonical argument for $((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)$ in basis C from premisses $\alpha_1 \ldots \alpha_n$. □

As is the case with Peirce's rule, Mint's rule is not derivable in natural deduction systems (without the classical rule for $\bot$). These localized failures of the verificationist justification procedure may pose serious problems to the philosophical program vindicated by Dummett. In his paper, Sandqvist (2009) took some steps towards questioning the widespread belief among anti-realists that the classical canons of reasoning presuppose the principle of bivalence. He showed how a certain constructive semantics would validate the double-negation elimination rule, thus justifying classical logic. Notwithstanding, he remarks (p. 215):

> "In endeavouring to cook a classical logic out of constructively kosher ingredients, I am not trying to establish that logic as the only justifiable one. In particular, I do not mean to suggest that the logical constants, as intuitionists construe them, are 'really' subject to classical laws of inference. Our treatment of conditionals, absurdity and universal quantification certainly differs in some respects from standard intuitionist accounts; that is how we managed to achieve classicality."

If our definitions are an appropriate statement to what is proposed by Dummett (1991), then we have reasons to believe that the criticisms apply very well, at least in the propositional level[10]. In fact, Sandqvist's clause for the absurdity constant $\bot$ is exactly that of Dummett (1991, p. 295). If double-negation elimination is valid under Dummett's semantics then, by defining the other classical constants with the help of $\supset$, $\wedge$ and $\bot$, we have a constructive account of classical reasoning, apparently without assuming bivalence.

[10]Some people will notice that our examples of validation of rules are not similar enough to Dummett's own examples. However, a careful examination, we believe, will show that Dummett's examples are, sometimes, in conflict with his definitions. It was necessary for us, then, to make a choice. We decided to bear with Dummett's definitions instead of trying to extract from the examples whatever was his actual intentions for the workings of the justifications procedures.

3.3 Pragmatism

Despite his emphasis on introduction rules and verificationism, Dummett believes we might as well take a pragmatist stance and consider elimination rules instead. So, just as he did with the introduction rules, he gives a justification procedure intended to validate all rules of inference given a set E of elimination rules. In this section, we carry our purpose of evaluating the validity of admissible rules in Dummett's semantics to this new justification procedure.

Definition 7. In an elimination rule for a given constant γ the sole premiss which is required to have γ is called the *major premiss*, all others, if there are any others, we call *minor premisses*. Such an elimination rule can be a *vertical rule*, when the conclusion of any of its minor premisses coincides with the conclusion of the rule, or *reductive* otherwise. We require that the minor premisses of a vertical rule, which we may call *vertical premisses*, actually depend on hypotheses discharged by the elimination rule. In case a minor premiss is not vertical we say it is *horizontal* (p. 283).

Example 3. We recall from our discussion in Chapter 1 that elimination rules express the deductive use of sentences as premisses from which we extract consequences. Some elimination rules are applicable only in the presence of a specific context represented by the minor premisses. The minor premisses can be deductions whose conclusion is also the conclusion of the rule, in which case, as stated in the definition, we call them vertical premisses. We reproduce some natural deduction elimination rules so as to illustrate the definition above.

$$\dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\ \vee\text{E} \qquad\qquad \dfrac{A \quad A \supset B}{B}\ \supset\text{E}$$

As we can see $\vee$E is vertical and, therefore, its two minor premisses are vertical premisses. In contrast, the elimination rule for $\supset$ is reductive and its minor premiss is horizontal.

Definition 8. We call the occurence of a sentence *principal* if all the sentences on the path from itself to the conclusion are either a major premiss of an elimination rules or a premiss of a basic rule. Also, if a principal sentence is premiss of a given argument, we call this argument *proper*. Furthermore, there is no horizontal premiss in the path from a *placid* sentence occurrence to the conclusion of the argument (p. 284).

Example 4. In the argument below, the occurrences of C as minor premisses of $\vee$E are not placid, since the conclusion of the rule **C** (in bold face) is also a horizontal minor premiss of $\supset$E.

$$\dfrac{\dfrac{A \vee B \quad \begin{matrix}[A]\\ \Sigma_1\\ C\end{matrix} \quad \begin{matrix}[B]\\ \Sigma_2\\ C\end{matrix}}{\mathbf{C}} \qquad C \supset D}{D}$$

Definition 9. A canonical argument has the following properties (p. 284):

(i) its conclusion is an atomic sentence

(ii) it is proper

(iii) the subargument for any placid vertical premiss is proper

Moreover, an argument which is not canonical and whose conclusion is a horizontal premiss is called a *critical subargument*.

The item (iii) of the definition above is intended to avoid vertical premisses of elimination rules which are not proper and whose conclusion are also major premiss of an elimination. Still, there is a procedure, familiar from the reduction of segments in normalization for intuitionistic logic, which can bring arguments violating item (iii) into canonical form. Thus, consider the argument of Example 4. In addition, suppose that the subargument for the vertical premiss of $\vee$E is not proper. It can be made proper by rearranging it as below.

$$\dfrac{A\vee B \quad \dfrac{\begin{matrix}[A]\\ \Pi_1\\ C\end{matrix} \quad C\supset D}{D} \quad \dfrac{\begin{matrix}[B]\\ \Pi_2\\ C\end{matrix} \quad C\supset D}{D}}{D}$$

After the transformation, the vertical premiss has $C \supset D$ as a principal premiss and so, by Definition 8, is proper. The same transformation can be iterated such that there is no loss of generality if we assume the conclusions of vertical premisses to be always atomic.

At some point in the discussion of the pragmatist justification procedure, Dummett occupies himself with canonical arguments whose critical subarguments have the same degree of the canonical arguments in which they occur. This forces him to make a distinction between validity in a broad and narrow sense, when applied to canonical arguments. Despite the distinction, Prawitz (2007, note 15) makes what we believe to be an unjustified criticism of Dummett's pragmatist justification procedure. He claims that the procedure will not recognize the general validity of *modus ponens*. He offers as a counterexample an application of *modus ponens* where the argument for the horizontal premiss proceed from the premiss of highest complexity in the whole argument. According to Prawitz, this particular instance of *modus ponens* will not be valid by Dummett's definitions since it violates the complexity requirement necessary to avoid circularity.

Now, with regard to Prawitz's criticisms, we would like to observe, first, that Dummett (1991, p. 284) consider explicitly an argument similar to Prawitz's own example. Second, the procedure can be shown to be well-founded even if some critical subargument Π_2 happen to have the same degree as the original argument Π_1 since, at most, the validity of Π_2 will eventualy depend on the validity of another argument Π_3 of lower complexity after the premiss with highest degree is taken as principal premiss. However, baffling as it is, Dummett (2007, p. 484) seems to accept Prawitz's criticisms when, in his reply, he writes: "I fully accept Dag Prawitz's correction of my handling of the notion of a valid argument when elimination rules are taken as basic".

Circularity is a danger when we consider some unusual elimination rules, especially ones that do not respect Dummett's complexity constraint. In most of the cases, however, there are no problems. Thus, we take for granted that the transformations eventually lower the degree of the critical subargument and move on with the definitions leaving futher discussion of the circularity problem aside.

Definition 10. The complementation of an argument results from the replacement of its conclusion A, if complex, by a valid canonical argument of which A is a principal premiss (p. 285).

In a complementation of an argument Π_1, when substituting a canonical argument Π_2 for the conclusion, the resulting argument Π_3 will have the same atomic conclusion as Π_2 and its premisses will be those of Π_1 together with those of Π_2. There would be no loss of generality, in case the set E contains the usual $\vee$E, to assume its conclusion to be atomic.

Sometimes, when complementing an argument we need additional premisses to use as minor premisses of eliminations. When they are not derivable from the original premisses of Π_1, the minor premisses of $\supset$I are taken as additional premisses of the complementation. Also, Dummett (1991, p. 285–286) assumes that the arguments for the vertical premisses C of an application of $\vee$E whose major premiss is $A \vee B$ are derived by $\supset$E with the help of additional premisses $A \supset C$ and $B \supset C$, where A and B are discharged as shown in the next example.

Example 5. The complementation of an argument from a sentence of the form $A \wedge B$ to conclusion $A \vee B$ must take the form shown below.

$$\dfrac{\dfrac{A \wedge B}{A \vee B} \qquad \dfrac{A \supset C \qquad [A]}{C} \qquad \dfrac{B \supset C \qquad [B]}{C}}{C}$$

After complementation, the premisses of the argument will be $A \wedge B$, $A \supset C$ and $B \supset C$. There is no loss of generality if we assume C to be atomic. Basic rules can be applied to C to obtain other atomic conclusions.

Definition 11. An arbitrary argument is *valid* if there is an effective method of finding, for any complementation of it, a valid canonical argument with the same premisses and conclusion (p. 286). And a canonical argument is valid if all its critical subarguments are valid (p. 284)

Example 6. Since the theorems below are dedicated to show invalidity, we provide a pragmatist justification for the inference of $A \supset \neg\neg B$ from $\neg\neg(\neg A \vee B)$ as an illustration. First, following Definition 10, we consider a complementation of the argument.

$$\dfrac{\dfrac{\dfrac{\neg\neg(\neg A \vee B)\text{①}}{A \supset \neg\neg B} \qquad A\text{②}}{\neg\neg B} \qquad \neg B\text{③}}{\bot}$$

We indicate the premisses of the complementation by circled numbers "①", "②", "③" and so on. By Definition 11, we have to obtain a valid canonical argument for the same conclusion $\bot$ from the same premisses. Any valid canonical argument, by Definition 9, is proper. This means that at least one of its premisses is principal, by Definition 8. We choose ① to be the principal premiss of our valid canonical argument and apply the corresponding elimination rule.

$$\dfrac{\neg\neg(\neg A \vee B) \quad \boxed{\dfrac{A \qquad \neg B}{\neg(\neg A \vee B)}}}{\bot}$$

By Definition 11, the canonical argument above would be valid provided its critical subargument, surrounded in a box, is valid. We repeat the process for the critical subargument.

$$\frac{\dfrac{A\,\textcircled{1} \qquad \neg B\,\textcircled{2}}{\neg(\neg A \vee B)} \qquad \neg A \vee B\,\textcircled{3}}{\bot}$$

This time, we choose ③ as the principal premiss of our valid canonical argument.

$$\frac{\neg A \vee B \qquad \dfrac{[\neg A] \qquad \boxed{\dfrac{A}{\neg\neg A}}}{\bot} \qquad \dfrac{[B] \qquad \neg B}{\bot}}{\bot}$$

The critical subargument from A to $\neg\neg A$ in the canonical argument above is obviously valid. And this concludes our example.

There are some noteworthy differences between the verificationist and the pragmatist justification procedures. By comparison, the pragmatist procedure is much more favorable to the view of deduction as proceeding from assumptions. Indeed, the justification procedure does not change essentialy if we simply drop the requirement for complementation to go all the way down to atomic conclusions. Also, there is no need to explicitly consider basic rules. In addition, we notice that, after the complementation, it rests on us the responsibility to show the validity of the critical subargument in the pragmatist justification procedure. This situation constrasts with the verificationist one, where supplementations, being valid canonical arguments, already gives us valid critical subarguments.

In what follows, we shall evaluate the validity of the same rules discussed in Section 3.2 with respect to the pragmatist justification procedure. In order to remain completely faithful to Dummett's definitions and also for the sake of uniformity, we maintain the assumption that the sentences are atomic in the following theorems.

Theorem 3. *Let φ and ψ be atomic sentences. Then, given any set B of production rules, Peirce's rule*

$$\frac{(\varphi \supset \psi) \supset \varphi}{\varphi}$$

is not valid with respect to $\supset E$.

Proof. Since the the conclusion φ is an atomic sentence, there is nothing to complement. In other words, an instance of Peirce's rule is its own complementation. Now, we need to ask whether is possible to obtain a valid canonical argument from $(\varphi \supset \psi) \supset \varphi$ to conclusion φ. Any valid canonical argument from premiss $(\varphi \supset \psi) \supset \varphi$, being proper, will have that premiss as a major premiss of $\supset$E (by Definition 8).

$$\frac{\varphi \supset \psi \qquad (\varphi \supset \psi) \supset \varphi}{\varphi}$$

Here, the argument for $\varphi \supset \psi$ is a critical subargument (again, by Definition 9). At this point, Peirce's rule would be valid only if we could infer $\varphi \supset \psi$ from no premisses. But this is not the case, since a complementation of $\varphi \supset \psi$, while having ψ as conclusion, will require φ as minor premiss. □

It seems that Dummett is safe from validating underivable rules in the context of eliminations and the pragmatist justification procedure. Next, we see what the pragmatist justification procedure says about Mints' rule.

Theorem 4. *Let φ, ψ and χ be atomic sentences. Then, given any set B of production rules, Mints' rule*

$$\frac{(\varphi \supset \psi) \supset (\varphi \vee \chi)}{((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)}$$

is not valid with respect to $\supset$E and $\vee$E.

Proof. Complementing Mint's rule we have to replace its conclusion by a valid canonical argument from $((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)$, as *principal* premiss, to atomic conclusions (by Definition 10). So, this argument will have $((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)$ as major premiss of $\vee$E. As illustrated in Example 5, we now have an argument with an atomic conclusion, say ω, and premisses $(\varphi \supset \psi) \supset (\varphi \vee \chi)$, $((\varphi \supset \psi) \supset \varphi) \supset \omega$ and $((\varphi \supset \psi) \supset \chi) \supset \omega$.

$$\frac{\dfrac{(\varphi \supset \psi) \supset (\varphi \vee \chi)}{((\varphi \supset \psi) \supset \varphi) \vee ((\varphi \supset \psi) \supset \chi)} \quad \dfrac{((\varphi \supset \psi) \supset \varphi) \supset \omega \quad [(\varphi \supset \psi) \supset \varphi]}{\omega} \quad \dfrac{((\varphi \supset \psi) \supset \chi) \supset \omega \quad [(\varphi \supset \psi) \supset \chi]}{\omega}}{\omega}$$

However, no matter which one of the premisses we take as principal, there is no way to obtain a valid canonical argument for conclusion ω appealing only to the same three premisses of the complementation. □

The sharp disagreement in the cases of Peirce's rule and Mints' rule is sufficient to show an unbalance between the verificationist and the pragmatist justification procedures with respect to the usual introduction and elimination rules for the propositional logical constants. In what follows, we explore some consequences of our investigations.

Conclusion

As we saw on Chapter 1, the notion of harmony between the uses of an expression has to do with two particular aspects. First, the conditions for the correct inference of sentences containing that expression. Second, what conclusion we can draw from the assumption of a sentence containing that expression. Those two aspects are harmonious if what we extract from a sentence is no more, nor less, than what we require for the correct inference of that same sentence. This relationship, when applying to rules governing the use of a single expression, are called *intrinsic harmony*. On the other hand, *total harmony* refers to the balance between the two aforementioned aspects when applied to a hole set of expressions, any number of them may occur in a single sentence.

Acknowledging that the intrinsic harmony between the rules for each expression is insufficient, Dummett (1991, p. 287–288) devised the requirement of stability as a criterion to evaluate total harmony for the inferential practice determined by some set of rules governing the logical constants. According to this criterion, if, given a set of introduction [elimination] rules, we get the same set after applying both justification procedures, then stability holds.

The only restriction that Dummett (1991, p. 258) places upon introduction and elimination rules is what he calls a complexity condition. For introduction rules, this means that the premisses and discharged hypotheses of the rule must be of lower complexity than the conclusion. On the other hand, Dummett (1991, p. 283) requires elimination rules to have the conclusion, minor premisses and discharged hypotheses with lower complexity than the major premiss.

In the previous chapter, we saw that, given $\supset$I, the verificationist justification procedure justifies Peirce's rule as an additional elimination rule (besides the usual *modus ponens*). Notwithstanding, Peirce's rule is not justified by the pragmatist justification procedure from $\supset$E. So, if we begin from a set E containing $\supset$E, after applying both justification procedures, we get another set which contains at least Peirce's rule as an additional elimination rule. Therefore, stability fails.

There are at least two possible ways to face the difficulty. We can accept the overall justificatory procedures given by Dummett and look for different introduction and elimination rules which are stable with respect to those procedures. Or we can try to reformulate Dummett's procedures, guided by the objective of giving a better account of hypothetical reasoning. Throughout our discussions, we have tentatively tryed to advance an alternative course along the second line. In this respect, we shall comment briefly on some positive facts that emerged from our investigations.

First, we call atention to the relative sucess of the pragmatist justification procedure. Besides seeming to work as expected, the pragmatist justification procedure has a clear algorithmic application. Also, on reflection, it is easy to see how to obtain, from the justification procedure with respect to the usual elimination rules, a normal derivation in natural deduction. The converse construction, however, seems more complicated. Such constructions can be used to provide some kind of semantical completeness and soundness arguments with respect to natural deduction (if one is interested in such results). On the other hand, we can not see, at present, a clear, general and algorithmic way to apply our corrected version of the verificationist justification procedure. Nonetheless, we hope that our discussion has thrown some light in the problems pertaining to the substitutional view of open proofs. Lastly, we sincerely expect that further investigations on these topics will prove fruitful enough for us to settle positively the question of stability.

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